L'Huilier's Theorem, Spherical Triangles, and the Radius of the Earth

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This is a recreational problem in spherical trigonometry for people who have never met L'Huilier's Theorem.[1] Consider the air distance map shown in Figure 1. For four cities (here Boston, Seattle, Los Angeles and Miami) all six pairwise distances are shown in statute miles.

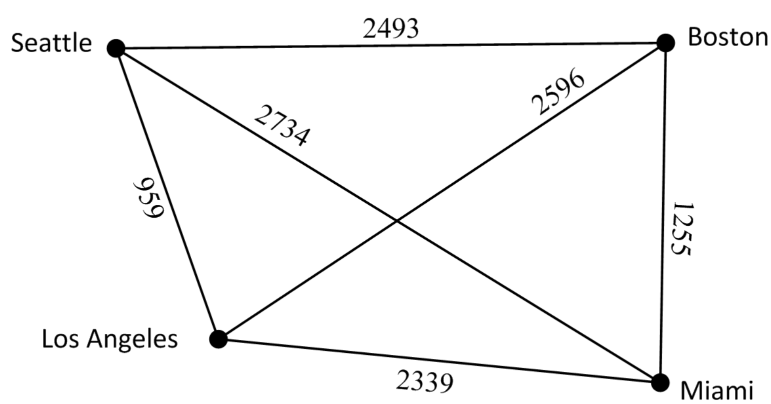

Figure 1: Air mileage between all pairs of four cities.

The question is, simply, what is the radius R of the Earth? Since this is only recreational, we'll take the Earth to be spherical.

It is obvious that we need at least four cities to answer the question. Three cities not enough, because, given any set of three distances, we can construct locations for them on a sphere of any (sufficiently large) radius: Put two cities at their specified distance and locate the third at the intersection of the circles, centered on the first two, with appropriate radii.

Although the question is simple, it is not so clear what set of trigonometric identities can lead to its solution. That is where L'Huilier's Theorem comes in. The reader is invited to try to solve the problem before reading further.

L'Huilier's Theorem

L'Huilier's Theorem gives the spherical excess E of spherical triangle in terms of its three sides a, b, and c. Recall that the spherical excess is

$$E \equiv \alpha + \beta + \gamma - \pi \tag{1}$$

where α, β, γ are the triangle's three angles. L'Huilier's Theorem is

$$\tan \tfrac{1}{4}E = \sqrt{\tan \tfrac{1}{2}s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)} \tag{2}$$

where

$$s \equiv \tfrac{1}{2}(a+b+c) \tag{3}$$

All quantities are measured in radians.

Also recall Girard's Theorem, that any triangle's spherical excess (in radians) is equal to its area (in steradians). (L'Huilier, by the way, lived from 1750 to 1840, while Girard's years were 1765 to 1836. Spherical trigonometry was once a hot topic!)

Problem Solution

The radius of the Earth, R, enters in converting distances from air miles to radians, and areas from steradians to square miles. L'Huilier's Theorem thus readily gives the area Q of a triangle on Earth, in square miles, in terms of its sides A, B, and C, in (air) miles,

$$Q = 4R^2 \arctan \sqrt{\tan[\tfrac{1}{2}S/R] \tan[\tfrac{1}{2}(S-A)/R] \tan[\tfrac{1}{2}(S-B)/R] \tan[\tfrac{1}{2}(S-C)/R]} \tag{4}$$

where

$$S \equiv \tfrac{1}{2}(A+B+C) \tag{5}$$

In Figure 1, we can express the area of the quadralateral as the sum of two spherical triangles in two different ways,

$$Q_{SBL} + Q_{BML} = Q_{SML} + Q_{SBM} \tag{6}$$

where S, B, M, L denote Seattle, Boston, Miami, and Los Angeles, respectively. While the common R^2 in each Q term cancels out, the remaining parts are a transcendental equation for R. This is easily solved numerically, as shown, for Mathematica, in Figure 2.

We obtain $R = 3991$ mi, slightly larger than both the actual equatorial radius of the Earth, 3963 mi, and the polar radius, 3949 mi. Presumably, with more accurate input data, we would obtain an answer that lies between the equatorial and polar values.

References

[1] Weisstein, Eric W. "L'Huilier's Theorem." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/LHuiliersTheorem.html

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Q[A_, B_, C_, R_] :=
 (S = (A + B + C) / 2;
  4 ArcTan[Sqrt[Tan[S / (2 R)] Tan[(S - A) / (2 R)]
      Tan[(S - B) / (2 R)] Tan[(S - C) / (2 R)]]])

Qsml = Q[2734., 2339., 959., R];
Qsmb = Q[2734., 2493., 1255., R];
Qlbm = Q[2596., 1255., 2339., R];
Qlbs = Q[2596., 2493., 959., R];

Plot[Qsml + Qsmb - Qlbm - Qlbs, {R, 3500, 4500}]
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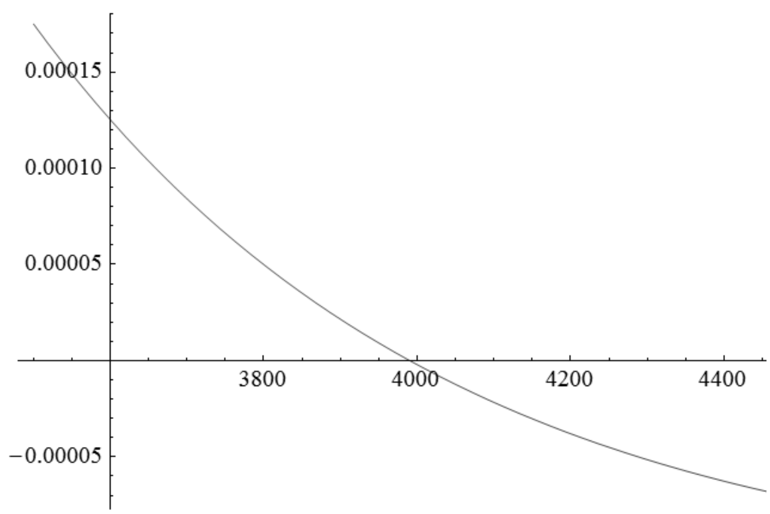


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FindRoot[Qsml + Qsmb - Qlbm - Qlbs, {R, 3900.}]

{R → 3990.54}
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Figure 2: Solution of the problem using L'Huilier's Theorem.