3 Radiation

Virtually all the information we get in astronomy comes to us as electromagnetic radiation (radio, infrared, visible, UV, X-ray, or gamma ray), in other words as photons.

3.1 Photon description of light

3.1.1 Photons

Light (or any EM radiation) comes in discrete particles called photons. The basic principles are

- they move in straight lines at the speed of light c,
- they carry energy E and momentum p (each photon has some particular energy E),
- they are conserved in number, *except* if emitted or absorbed by a charged particle (e.g. the electron in an atom).

Each photon's E and p are related by

$$E = pc \; .$$

Note that p is the magnitude of the 3-vector momentum $\boldsymbol{p}$

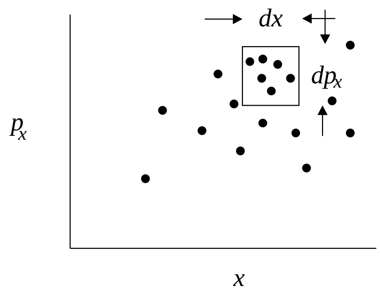
$$p = |\boldsymbol{p}| \; .$$

3.1.2 Phase space density

A big part of understanding radiation is just bookkeeping on the positions and directions of the photons: it's not just the number (density) of photons at each point that matters, but also the distribution of which way they are going, and of what energies they have. Since the momentum vector $\boldsymbol{p}$ encapsulates both the latter concepts (direction and energy) it is useful to think, at some instant in time, of each photon being at a point in a 6-dimensional space of position and momentum:

$$(x, y, z, p_x, p_y, p_z) \ .$$

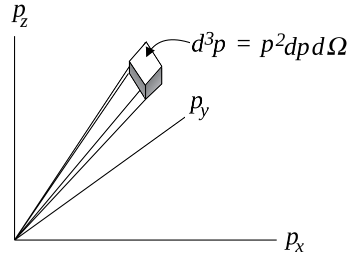
(I can only draw 2 of the dimensions:)



The phase space density $\mathcal{N}$ is the number of photons dN per 6-dimensional unit phase space:

$$dN = \mathcal{N}\, dx\, dy\, dz\, dp_x\, dp_y\, dp_z \equiv \mathcal{N}\, d^3\boldsymbol{x}\, d^3\boldsymbol{p} \ .$$

Things get more interesting if we decide to keep track of the momentum part of phase space in spherical, not Cartesian coordinates.



$d\Omega$ is a solid angle

Then,

$$d^3\boldsymbol{p} = p^2\,dp\,d\Omega = \frac{1}{c^3}E^2 dE\,d\Omega\ .$$

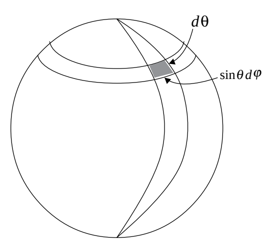
So, for photons in a fixed energy interval dE around E, d^3p is proportional to the solid angle element $d\Omega$ that encompasses their directions and to $E^2 dE$.

We should here say a word about the notation $d\Omega$ for solid angle; it is measured in steradians, a unit such that the full 2-sphere is 4π steradians, i.e.

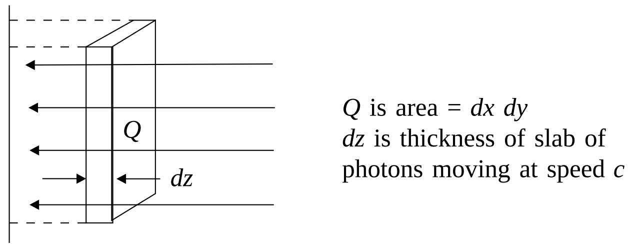
$$\int_{\text{sphere}} d\Omega = 4\pi\ .$$

In spherical polar coordinates, $d\Omega$ decomposes into the two orthogonal directions,

$$d\Omega = \sin\theta d\theta d\varphi = -d(\cos\theta)d\varphi\ .$$



If we consider photons moving in one direction and *normally incident* on a detector, then d^3x is related to their flux:



Photons in volume $dx\,dy\,dz$ will slam into wall in a time dz/c. So,

$$\text{(number per volume)} = \frac{1}{c}\,\text{(number per area per time)},$$

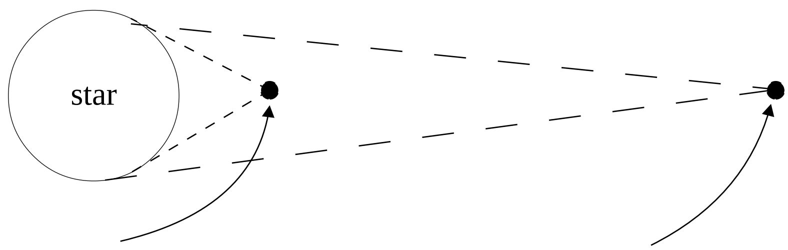
or

$$d^3x = c \underbrace{dx\,dy}_{\substack{\text{normal area} \\ d^2\Sigma}} dt \equiv c\, d^2\Sigma\, dt\ .$$

Why do we care? Because of a very famous, subtle theorem of mechanics:

Liouville's Theorem: *Phase space density of particles is* constant *along the trajectory of any freely moving particle.*

Example: Compare photons close to and far from an extended source like a star:



So, dilution of photon density is always accompanied by increased collimation in a precise quantitative relationship!

3.1.3 Brightness

Let's compute the total energy $d\mathcal{E}$ in a small volume of phase space

$$\begin{aligned} d\mathcal{E} = E dN &= E\mathcal{N} d^3x\, d^3p \\ &= E\mathcal{N}(cd^2\Sigma\, dt)\left(\frac{1}{c^3}E^2\, dE\, d\Omega\right) \\ &= \left(E^3\mathcal{N}/c^2\right) d^2\Sigma\, dt\, dE\, d\Omega \\ &\equiv I d^2\Sigma\, dt\, dE\, d\Omega\ . \end{aligned}$$

Here we have defined

$$I = \mathcal{N}E^3/c^2\ .$$

We see that I is the energy in photons *per (normally incident) area per time per solid angle per photon energy interval dE.* I is called the "specific intensity" or "brightness" of a radiation field at a particular point, in a particular direction, at a particular energy.

Note that since $\mathcal{N}$ is conserved (Liouville) and E is constant for each photon, I is conserved too! If you look at a source while moving farther and farther away, its brightness (in this context often called "surface brightness") doesn't change. It gets smaller but not dimmer (per unit apparent size).

All optical systems indeed satisfy Liouville's Theorem. One implication is that no telescope, however large, can make a scene look *brighter* than it looks to the naked eye. It can only make it look *bigger*. However, for very small "unresolved" sources like stars, bigger effectively *is* brighter, since there is more light in the resolution element.

CCD detectors and photographic emulsions don't measure surface brightness (as the eye does), but rather respond to total energy deposited per area of collector. For these, a bigger telescope, gathering more total photons, *is* advantageous. We will come back to this topic later.

3.2 Wave description of light

3.2.1 Waves

Because of quantum mechanics "wave-particle duality," photons also can exhibit wave-like properties. In fact, in the classical limit of many photons, they are described by the classical Maxwell equations of electromagnetism.

Basic principles:

- Relation between frequency ν and wavelength λ:

$$\nu = \frac{c}{\lambda}$$

- Two linear polarizations
- Reflection from conducting surfaces (e.g. the aluminized surface of a telescope mirror)

- Refraction in lenses by the "principle of least time"

- Interference effects

- Finite resolution of an aperture: A telescope of aperture (diameter) D cannot distinguish directions any finer than about

$$\delta\theta = \frac{\lambda}{D} \quad \text{(in radians)}$$

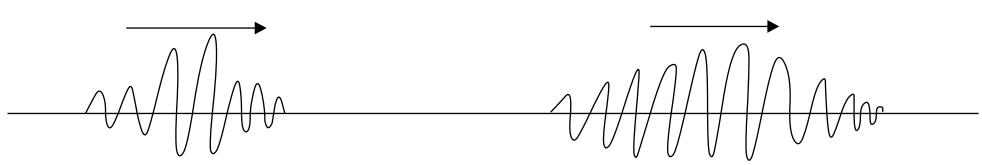
The light from unresolved "point" sources is seen as blurred into a spot of about this radius.

3.2.2 Connection between particle and wave descriptions

The energy of a photon (a quantum concept) is related to the frequency of the photon (a classical concept) by Planck's famous

$$E = h\nu$$

where $h = 6.63 \times 10^{-27}$erg sec is Planck's constant. One way to visualize the photon as both a particle and wave is to think of it as a propagating wave packet.



The more monochromatic (small ΔE or $\Delta\nu$) the photon, the longer the wave packet. This is why it is easy to do interference demonstrations with laser light (very monochromatic) but quite difficult with white light (large $\Delta\nu$). The relation between length of wave packet and frequency (or energy, or momentum) localization is of course the Heisenberg uncertainty principle.

Notice that

$$dE = h d\nu \ .$$

3.3 Radiation units

3.3.1 Specific intensity I_ν

It is more common to characterize light intensity on a "per frequency" basis than on a "per photon energy" basis. In terms of our old I (brightness) we define

$$I_\nu \equiv I \frac{dE}{d\nu} = hI \ .$$

Then, the first two equations in 3.13 above become

$$\text{phase space density } \equiv \mathcal{N} = \frac{I_\nu}{\nu^3} \frac{c^2}{h^4}$$

and

$$d\mathcal{E} = I_\nu d^2\Sigma \, dt \, d\nu \, d\Omega$$

with I_ν having the units energy per (normal incidence) area per time per frequency interval per solid angle, that is, erg s^{-1} cm^{-2}ster^{-1}Hz^{-1}.

I_ν is a function of:

- position in space
- direction under consideration
- photon frequency under consideration
- (possibly, though not usually) time

3.3.2 Specific intensity I_λ

Occasionally one wants to do the energy bookkeeping in terms of wavelength λ instead of frequency ν. On then *defines* I_λ by the relation

$$|I_\nu \, d\nu| = |I_\lambda \, d\lambda|$$

So that $d\mathcal{E} = I_\lambda \, d^2\Sigma \, dt \, d\lambda \, d\Omega$. (The absolute value bars are to be sure that energy comes out positive.) Using $\nu = c/\lambda$,

$$\left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2} = \frac{\nu^2}{c} \; .$$

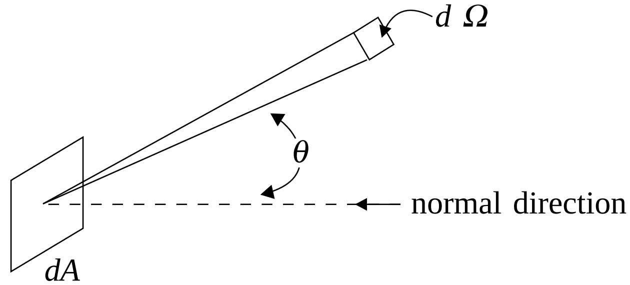
So

$$I_\lambda = \frac{\nu^2}{c} I_\nu = \frac{c}{\lambda^2} I_\nu \; .$$

Do not be confused by these extra factors of ν or λ. They are just bookkeeping to make "per frequency" and "per wavelength" come out consistently when describing the same amount of energy.

3.3.3 Net flux

If we want to know the total energy deposited onto a specific element of area dA (on a detector, say), we integrate up the specific intensity coming from solid angle elements in all directions. However, they do not all get equal weight: the farther they are from normal incidence, the less "projected area" of detector is available to them.



The net flux F_ν (units erg $\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\mathrm{Hz}^{-1}$) is thus

$$F_\nu = \int I_\nu \cos\theta \, d\Omega \ .$$

In spherical coordinates, $d\Omega = \sin\theta \, d\theta \, d\varphi = -d\cos\theta \, d\varphi$. What are the limits of integration? If dA represents an opaque detector, θ goes from 0 to 90°. On the other hand, if dA is transparent, then θ goes from 0° to 180°, and the $\cos\theta$ weighting is negative on the back side ($90° < \theta < 180°$).

For an isotropic radiation field I_ν is not a function of θ or φ, and one sees that the net flux (including from the back) is exactly zero.

As before we can do the bookkeeping "per wavelength" instead of "per frequency," with

$$F_\lambda \equiv \frac{\nu^2}{c} F_\nu = \frac{c}{\lambda^2} F_\nu \ .$$

F_λ is the energy flux per area per time per wavelength interval. In fact, it is exactly the same quantity that we called f_λ in the table of UBV photometry in 2.2.1.

Note that we are always free to integrate up all the different frequencies and obtain total integrated flux:

$$\begin{aligned} F &= \int F_\nu \, d_\nu = \int F_\lambda \, d\lambda \qquad (\text{erg s}^{-1} \text{ cm}^{-2}) \\ I &= \int I_\nu \, d_\nu = \int I_\lambda \, d_\lambda \qquad (\text{erg s}^{-1}\text{cm}^{-2}\text{ster}^{-1}) \end{aligned}$$

3.3.4 Energy density and radiation pressure

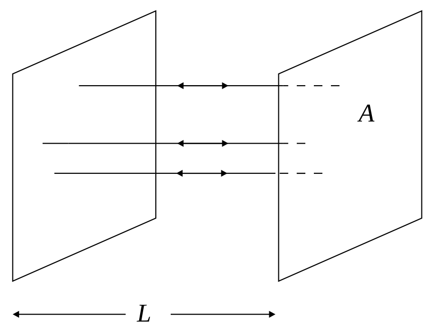
We already saw (3.1.2) that energy *flux* in a given direction is related to the energy *density* of photons moving in that direction by a factor of c: the block of photons smashes into the detector at that speed. Integrating over all directions, the total energy density at a point, due to photons with energies between ν and $\nu + d\nu$, is

$$u_\nu \, d\nu = \frac{1}{c} \left[\int I_\nu \, d\Omega \right] d\nu$$

and the total energy density is

$$u = \int u_\nu \, d\nu = \frac{1}{c} \iint I_\nu \, d\nu \, d\Omega \; .$$

Pressure is force per area, or momentum transfer per area per time. Consider photons bouncing back and forth between parallel perfect mirrors of area A and separation L.



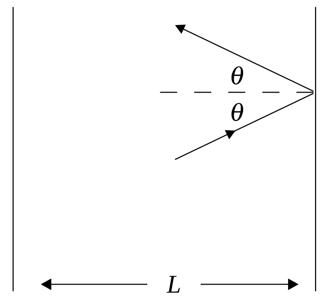
Each photon contributes $\Delta u = E/(AL)$ to the energy density (since the volume is AL). Each photon gives a momentum transfer $2p$ each time it bounces off the mirror ($+p$ goes to $-p$, so the difference is $2p$). These bounces occur (on a given mirror) every $2L/c$ time, so the photon's contribution to the pressure is

$$\Delta P = \frac{2p}{A(2L/c)} = \frac{pc}{LA} = \frac{E}{LA} = \Delta u \; .$$

Adding up all the photons, we would find

$$P = u \quad \text{(but wait!)}$$

We have forgotten that photons are moving at different angles. A photon moving at angle θ to the normal still contributes $E/(AL)$ to Δu, but



the momentum transfer is now only $2p\cos\theta$, and also it takes longer between bounces,

$$\Delta t = \frac{2L/C}{\cos\theta} \ .$$

Therefore

$$\Delta P = \cos^2\theta \Delta u \ .$$

Adding up all the photons is the same as averaging an isotropic distribution over $\cos^2\theta$:

$$\begin{aligned} \langle \cos^2\theta \rangle &= \frac{\iint \cos^2\theta \, d\theta \, \sin\theta d\varphi}{\iint d\theta \, \sin\theta \, d\varphi} \\ &= \frac{2\pi \int_0^\pi \cos^2\theta \, \sin\theta \, d\theta}{2\pi \int_0^\pi \sin\theta \, d\theta} = \frac{\int_{-1}^1 \mu^2 d\mu}{\int_{-1}^1 d\mu} = \frac{1}{3} \end{aligned}$$

(where we set $\mu = \cos\theta$). Another way to get this result is by symmetry: Consider a randomly oriented unit vector, (x, y, z), with $x^2 + y^2 + z^2 = 1$. Since there is nothing special about the x, y, or z directions with respect to the unit sphere, we must have

$$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle \ .$$

But

$$\langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = \langle x^2 + y^2 + z^2 \rangle = \langle 1 \rangle = 1 \ ,$$

so each one must be 1/3. Note that in spherical coordinates $z = \cos\theta$, and we are done.

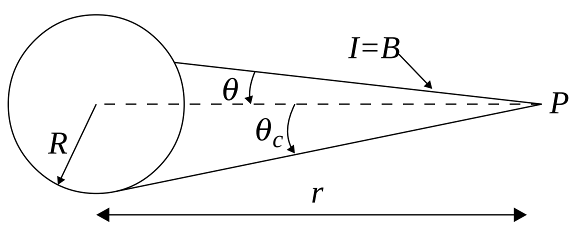
In either case, our final result is

$$P = \frac{1}{3} u \ .$$

The pressure of isotropic radiation is exactly 1/3 its energy density.

3.3.5 Example: Sphere of uniform brightness [From Rybicki and Lightman]

Let us calculate the flux at an arbitrary distance from a sphere of uniform brightness $I = B$ (that is, all rays leaving the sphere have the same brightness). Such a sphere is clearly an isotropic source. At P, the specific intensity is B if the ray intersects the sphere and zero otherwise.



Then,

$$F = \int I \cos\theta \, d\Omega = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta \cos\theta \, d\theta \ ,$$

where $\theta_c = \sin^{-1} R/r$ is the angle at which a ray from P is tangent to the sphere. It follows that

$$F = \pi B (1 - \cos^2\theta_c) = \pi B \sin^2\theta_c$$

or

$$F = \pi B \left(\frac{R}{r}\right)^2 \ .$$

Thus the specific intensity is constant, but the solid angle subtended by the given object decreases in such a way that the inverse square law is recovered.

A useful result is obtained by setting $r = R$:

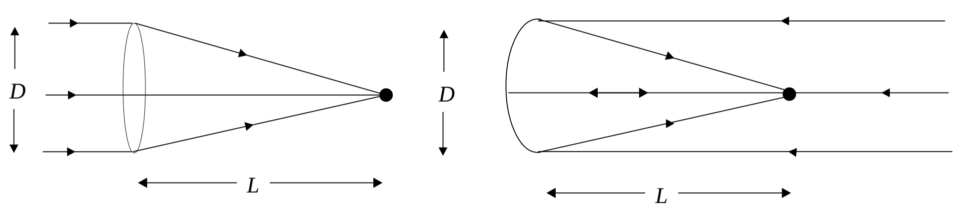
$$F = \pi B \ .$$

That is, the flux at a surface of uniform brightness B is simply πB.

3.4 Telescopes

3.4.1 Astronomical Telescopes

All modern telescopes (optical, radio, X-ray) are mirrors, not lenses. However, mirror optics is just lens optics folded over:

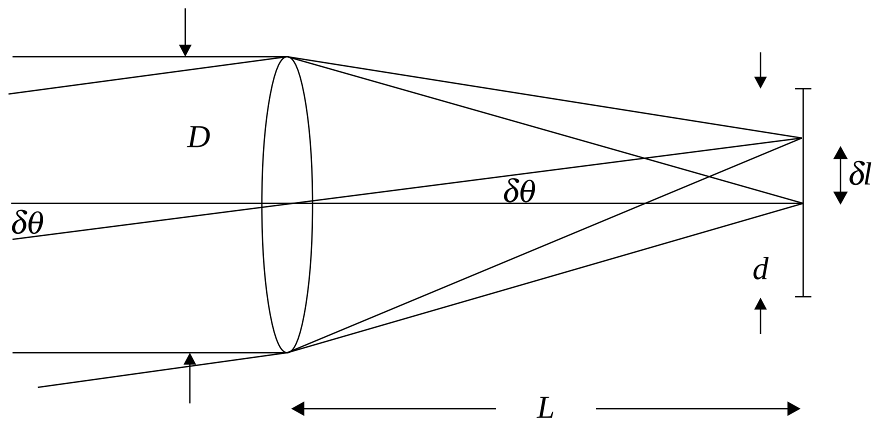


Since it is easier to draw lenses, we will use the left hand figure, but you should mentally translate it to the right-hand one. D is the aperture, L the focal length.

The basic rules of ray tracing are:

- rays go through center of lens undeflected
- parallel rays get converged and come together at the focal length

That is all we need to get:



This is the basic "prime focus" astronomical telescope. The detector (CCD array) goes at the focus. The "image scale" is the ratio of physical distance at the focus to angular distance in the sky, and it depends *only on the focal length* (see figure),

$$\frac{\delta l}{\delta\theta} = L \quad \text{or} \quad \frac{\delta\theta}{\delta l} = 1/L \ .$$

For example, the 16.8 m focal length of the Palomar 200 inch corresponds to 12 arcsec/mm (check this!)

The field of view is determined by the focal length and the size of the detector (d in the figure)

$$\Delta\theta_{FOV} = d/L \ .$$

In the above example, a CCD array of diameter 60 mm = 6 cm would have a field of view of

$$60\,\text{mm} \times \frac{12\,\text{arcsec}}{1\ \text{mm}} \times \frac{1\,\text{arcmin}}{60\,\text{arcsec}} = 12\,\text{arcmin} = 0.2^\circ \ .$$

What is the relation between the surface brightness of an object in the sky (a planet or galaxy, say) and the flux (energy per area per time) in its image on the detector? We multiply I_ν by the area of the lens (mirror) and by the *area conversion factor* from sterradians to cm^2 on the detector:

$$F_\nu\,(\text{detector}) = I_\nu\,(\text{source})\,\frac{\pi D^2}{4}\left(\frac{\delta\theta}{\delta l}\right)^2 = \frac{\pi}{4} I_\nu \left(\frac{D}{L}\right)^2 .$$

The quantity L/D is called the f number of the telescope, written "$f/\#$," so

$$F_\nu = \frac{\pi}{4} I_\nu / (f/\#)^2 .$$

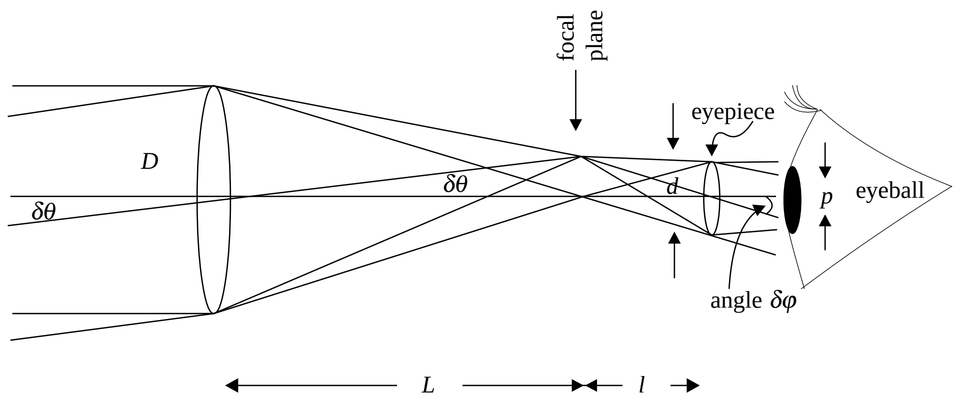
Notice that the image plane flux does not depend on the size of the telescope, but only on its $f/\#$. Photographers are familiar with this fact: a light meter dictates an $f/\#$ and shutter speed, independent of the focal length of the lens.

The Palomar 200″ is a "slow" telescope, about $f/3.5$ (that is, $f/\# = 3.5$). Modern telescopes are often between $f/1.75$ and (state of the art) $f/1.25$. That is, they are "short" (focal length) and "wide" (aperture). If image flux depends only on f-number, why not make tiny little fast ($f/1.25$) telescopes — Palomar in your pocket or Keck in your khakis? Answers:

- For bright objects (e.g. planets) we want the large image scale to get more pixels (resolution elements on the detector) across the image.
- For faint objects (e.g. distant galaxies) we are photon-starved and care not so much about energy per unit area on the focal plane as about total number of photons from the object per time, which scales as D^2.

3.4.2 Telescopes you look through (e.g. binoculars)

This is not astronomy, but it *once* was, and you might be interested.



D is the aperture; L is the focal length of the objective lens; d is the exit pupil; l is the eyepiece focal length

The basic idea is that the objective makes an image at the focal plane, and the eyepiece is simply a "magnifying glass" for looking at that image.

From the figure you can derive

$$M = \text{ magnification } = \frac{\delta\varphi}{\delta\theta} = \frac{L}{l} \ .$$

Notice that it is useless to have d bigger than p (the pupil size of the human eye, no greater than about 9 mm when dark adapted) because the extra rays simply do not enter the eye! Tracing the rays back, the maximum useful size of D is

$$D = Md \approx Mp \ .$$

Bigger than this does *not* make a brighter image (Liouville's theorem, again). Smaller than this *does* make a dimmer image. How can this be true without violating Liouville? Answer: for ray bundles of diameter smaller than p, the eye is no longer a good approximation to a "surface brightness meter" — it is artificially "stopped down." (In bright light, of course, the eye naturally stops down on its own, when the pupil contracts.)

3.5 Thermal ("Black Body") Radiation

3.5.1 Black Body radiation is universal

There are many different specific processes that can produce radiation (energy level changes in atoms, for example). These different processes can and do produce different spectra, that is, different functional forms for the dependence of I_ν on ν. It is therefore an amazing fact that all systems *in thermal equilibrium* at a temperature T produce the same *universal* spectrum, called the Planck or black body spectrum

$$I_\nu = B_\nu(T) \; .$$

The reason for this involves both some statistical mechanics and some quantum mechanics. Roughly it is that the electromagnetic degrees of freedom of the system come to the same thermal equilibrium as the mechanical (motion of particles) degrees of freedom, each one getting a mean energy of

$$\left(\begin{array}{c} \text{energy in each} \\ \text{degree of freedom} \end{array} \right) = \frac{1}{2} kT$$

where T is the temperature and $k = 1.3806 \times 10^{-16}\,\text{erg}/^\circ\,\text{K}$ is Boltzman's constant (really just a unit conversion to go from the historical but arbitrary temperature unit of "degrees" to the natural temperature unit of ergs).

The huge importance of the Planck spectrum in astrophysics is because many things we see are perfectly — or very nearly — in thermal equilibrium. So the Planck spectrum is almost the universal spectrum of astrophysics.

Example 1: The surface of the Sun is nearly a black body emitter at 5770° K.

Example 2: Looking at the "dark" night sky, we are actually looking at the cooling remnant radiation of the big bang — virtually a perfect black body at $T = 2.73°$ K.

3.5.2 Derivation of the Planck spectrum

In other courses you will learn to count the degrees of freedom (modes) in the electromagnetic field and derive the Planck spectrum.

However, we can jump to the answer using things we already know, if I just tell you a couple of "predigested" results from quantum mechanics and statistical mechanics.

You have probably already seen the so-called "Boltzmann factor" that gives the relative probability of something acquiring an energy E when it has a temperature T,

$$\text{probability} \propto e^{-E/kT} \ .$$

For example, this is used to get the density law for an isothermal atmosphere by putting $E = m_A g h$ (m_A is the mass of an air molecule),

$$\rho \propto e^{-m_A g h/kT} \ .$$

The Boltzman factor $e^{-E/kT}$ is actually the *classical approximation* (valid for the atmosphere problem) to a deeper quantum mechanical result which we now describe.

Remember *phase space density*, the number of particles per unit $d^3\boldsymbol{x}\, d^3\boldsymbol{p}$? In quantum mechanics there is a natural unit (a kind of "quantum") for phase space volume, and it is Planck's constant, cubed!

$$\left(\begin{array}{c} \text{one quantum unit} \\ \text{of phase space} \end{array} \right) = h^3 \ .$$

The *occupation number* of a quantum system is the number of particles in one quantum unit of phase space, or in terms of the phase space density $\mathcal{N}$

$$\text{occupation number} = \mathcal{N} h^3 \ .$$

The deep result on thermal systems is that they have a universal mean occupation number given by

$$\text{occupation number} = \frac{1}{e^{E/kT} \pm 1} \ .$$

Notice that if $E \gg kT$ the ± 1 is negligible and we recover the classical result. There are two fundamentally different kinds of particles in the universe

Fermions	obey Pauli exclusion principle includes protons, neutrons, electrons	use $+1$ above
Bosons	"like to clump" can correspond to classical wave fields include photons, gravitons	use -1 above

Now lets put the pieces together, using the result in 3.3.1 that relates $\mathcal{N}$ and I_ν

$$\begin{aligned} B_\nu(T) = I_\nu &= \frac{\mathcal{N} \nu^3 h^4}{c^2} = (\text{occupation number}) \ \frac{h\nu^3}{c^2} \\ &= 2\frac{h\nu^3}{c^2}\frac{1}{e^{E/kT} - 1} = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \ . \end{aligned}$$

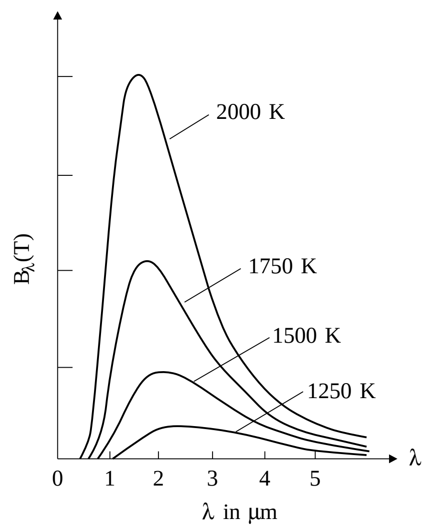
Where did the extra factor of 2 come from in the last two equations? It is the *two polarizations* of photons. Each one separately gets the occupation number $(e^{E/kT} - 1)^{-1}$, and I_ν counts both of them!

As before (Section 3.3.2) we have the option of accounting for the intensity on a per-wavelength instead of per-frequency basis,

$$B_\lambda = \frac{\nu^2}{c} B_\nu = \frac{c}{\lambda^2} B_\nu \ .$$

Since wavelength is easier to measure for light than frequency, one more frequently sees B_λ and I_λ in observational contexts. A graph of $B_\lambda(T)$ shows that hotter T's indeed give both

- overall greater intensity everywhere, and
- peak at shorter wavelengths.



3.5.3 Asymptotics of the Planck spectrum

For $h\nu \gg kT$, there is an *exponential* fall-off in brightness. This is called the "Wien limit." It is why you cannot get a tan under an ordinary incandescent light-bulb: The temperature of its filament is around 2700° K, and so it makes almost no UV.

For $h\nu \ll kT$ the exponential can be expanded

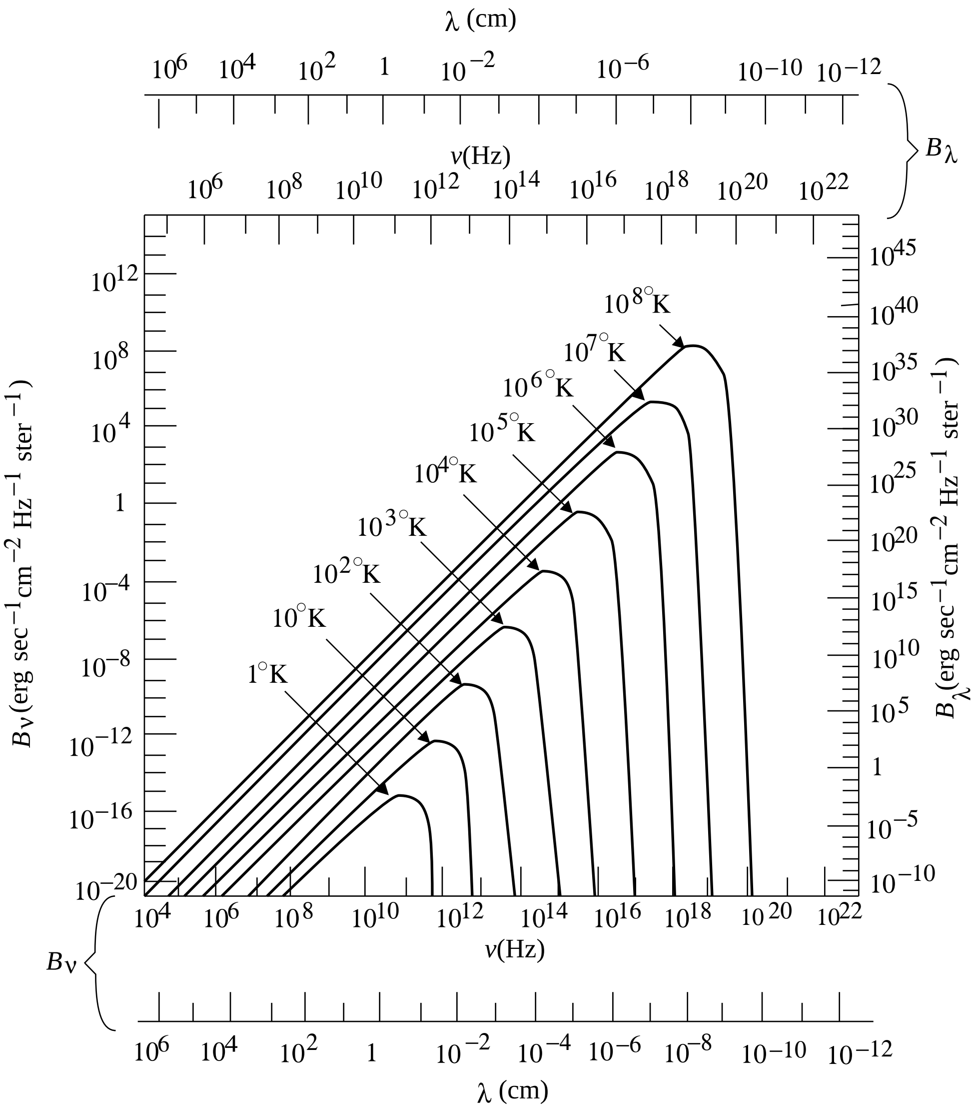
$$\exp\left(\frac{h\nu}{kT}\right) - 1 = \frac{h\nu}{kT} + \cdots$$

so that for $hv \ll kT$, we have the *Rayleigh-Jeans law:*

$$B_\nu(T) \approx \frac{2\nu^2}{c^2} kT \ .$$

Notice that this result does not contain Planck's constant. It was originally derived by assuming a mean energy $E = kT$, the classical equipartition value for the energy of an electromagnetic wave ($\frac{1}{2}kT$ in each of two polarizations). A plot of the Planck function over many decades of frequency, I_ν, and temperature shows clearly the asymptotic regimes (see figure on next page).

You can see why things are called "red hot" when the Wien limit is just barely poking into the red from the infrared; then "white hot" when the peak of $B_\nu(T)$ is at the eye's maximum visibility (about 6000° K); then "blue hot" when the eye's sensitivity is completely in the Rayleigh-Jeans regime. There is nothing "hotter than blue hot," because higher temperature just increases the Rayleigh-Jeans spectrum's amplitude, linearly with T, without changing the functional form of its ν^2 dependence on frequency.



The peak of the black-body spectrum is found, letting $x = h\nu/kT$, as follows:

$$
\begin{aligned}
0 &= \frac{d}{dx}\left(\frac{x^3}{e^x-1}\right) = \frac{3x^2(e^x-1) - x^3 e^x}{(e^x-1)^2} \\
&\Rightarrow \quad x = 3(1-e^{-x}) \\
&\Rightarrow \quad \text{(e.g. by trial and error) } x \approx 2.82\,,
\end{aligned}
$$

so at the peak

$$h\nu_{\max} = 2.82kT$$

or

$$\frac{\nu_{\rm max}}{T} = 5.88 \times 10^{10}\,{\rm Hz/\,K}\;.$$

Notice that the extra powers of λ in

$$B_\lambda(T) = B_\nu(T)\left|\frac{d\nu}{d\lambda}\right| = \frac{c}{\lambda^2}B_\nu(T)$$

shifts its peak a bit relative to B_ν, so at the maximum of B_λ (you can check this)

$$h\left(\frac{c}{\lambda_{\rm max}}\right)/kT = 4.97$$

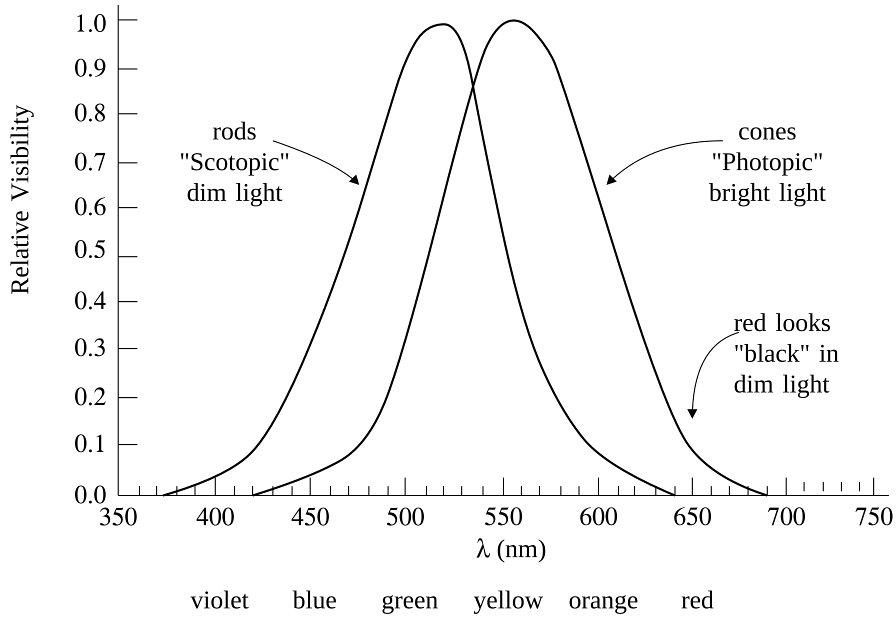
or

$$\lambda_{\rm max}T = 0.290\,{\rm cm\,K} \quad (\text{“Wien's displacement law.”})$$

The spectral sensitivity of the eye is different to bright light (cones) than to dim light (rods). Bright sensitivity peaks at 5600 Å(yellow) with an effective width of about 1100 Å. (Note close similarity to the V band in UBV!) Dim light sensitivity peaks at about 5000 Å(green) with about the same effective width (see the figure on the next page).

$$\begin{aligned} T &= \frac{(0.290\,{\rm cm\,K})}{5600{\rm Å}} = 5200\,{\rm K} \\ T &= \frac{(0.290\,{\rm cm\,K})}{5000{\rm Å}} = 5800\,{\rm K} \quad (\text{very close to } T \text{ of sun.}) \end{aligned}$$

Not coincidence that we evolved to put our peak sensitivity at peak of solar output! (Dim caves, moonlight, etc.)



3.5.4 Integral of the Planck spectrum: Stephan-Boltzman law

What is the total energy density in an isotropic radiation field at constant temperature (e.g. inside a furnace, or a star)?

From 3.3.4 we calculate

$$\begin{aligned} u &= \frac{1}{c}\iint I_\nu \, d\nu \, d\Omega = \frac{4\pi}{c}\int I_\nu d\nu \\ &= \frac{4\pi}{c}\int \frac{2h\nu^3/c^2}{e^{h\nu/kT}-1}\, d\nu \\ &= \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3\, dx}{e^x - 1} \end{aligned}$$

where $x \equiv h\nu/kT$. The integral is just a number . You can do it numerically, or look it up in a table of integrals. (Some day you should learn how to do it by *contour integration.*)

The answer is $\pi^4/15$, so

$$u = \frac{8\pi^5}{15}\frac{k^4}{h^3c^3}T^4 \equiv aT^4$$

where a is called the "radiation density constant" and has the value

$$a = 7.56464 \times 10^{-15} \, \mathrm{erg\, cm^{-3}\, K^{-4}} \; .$$

Next: What is the emergent *flux* from a black body surface? That is, how many ergs per second does it emit per unit area? In 3.3.5 we already obtained the result

$$F = \pi B$$

relating flux and brightness, so

$$\begin{aligned} F &= \pi \int I_\nu \, d\nu = \frac{c}{4} u \quad \text{(from above)} \\ &= \frac{ac}{4} T^4 \equiv \sigma T^4 \; . \end{aligned}$$

The physical constant $\sigma = ac/4$ is called the Stefan-Boltzmann constant and has the value

$$\sigma = 5.66956 \times 10^{-5} \, \mathrm{erg\, cm^{-2}\, s^{-1}\, K^{-4}} \; .$$

Example 1: A sphere of radius 7.0×10^{10} cm has a temperature of 5770° K. What is its luminosity?

$$\begin{aligned} L &= 4\pi r^2 F = 4\pi r^2 \sigma T^4 \\ &= 4\pi (7.0 \times 10^{10}\, \mathrm{cm})^2 (5770\, \mathrm{K})^4 (5.67 \times 10^{-5}\, \mathrm{erg\, cm^{-2}\, s^{-1}\, K^{-4}}) \\ &= 3.9 \times 10^{33}\, \mathrm{erg/\, s} \; . \end{aligned}$$

Example 2: How many watts of total luminous power are emitted by the standard "candle" that defines the lumen ($1/60\pi$ cm^2 at 2044° K)?

$$\begin{aligned} L = A\sigma T^4 &= \frac{1}{60\pi}\, \mathrm{cm^2} (2044\, \mathrm{K})^4 (5.67 \times 10^{-5}\, \mathrm{erg\, cm^{-2}\, s^{-1}\, K^{-4}}) \\ &= 5.25 \times 10^6\, \mathrm{erg/\, s} = 0.525\, \mathrm{W} \; . \end{aligned}$$

In a real candle, many times this much power, perhaps ~ 100, is carried off *convectively* by hot air. Note that at $T = 2044\,\mathrm{K}$ (melting point of platinum)

$$\lambda_{\max} = \frac{0.290\,\mathrm{cm\ K}}{2044\ \mathrm{K}} = 14000\mathring{A} \quad \text{(infrared!)}$$

So the “candle” is extremely red — we are seeing only its Wien-limit tail. Lumens and lux are so-called “photometric” (apparent) units, which therefore implicitly include the human eye’s sensitivity, as opposed to “radiometric” units, which are in absolute c.g.s. At more visible colors there are therefore many more lumens per watt than the value ≈ 2 that we got above for the dull red glow of melting platinum. At 5550 Å, there is a peak of 680 lumens per watt. The lumens per watt conversion at other wavelengths is this value times a standardized form of the sensitivity curve shown above (3.5.3). Light bulbs are often labeled with their lumens value as well as their wattage. For example, a typical 100W bulb may yield 1710 lumens for 750 hours.

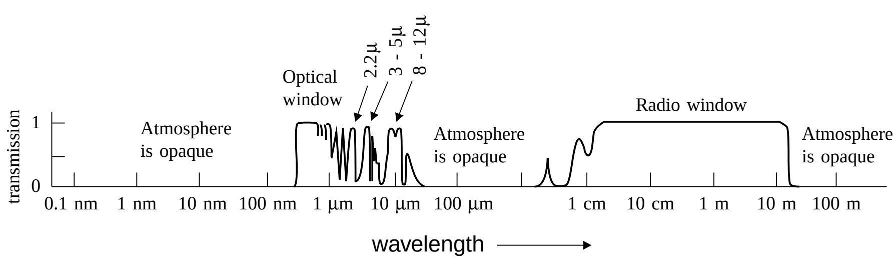
3.5.5 Color temperature and brightness temperature

Color Temperature. Often a spectrum is measured to have a shape more or less of blackbody form, but not necessarily of the proper absolute value. For example, by measuring F_ν from an unresolved source we cannot find I_ν unless we know the distance to the source and its physical size. By fitting the data to a blackbody curve without regard to vertical scale, a *color temperature* T_c is obtained. Often the “fitting” procedure is nothing more than estimating the peak of the spectrum and applying Wien’s displacement law to find a temperature.” [Rybicki and Lightman]

For example, the color temperature of moonlight is very nearly the same as that of sunlight. Long-exposure color pictures by moonlight appear in normal colors, despite the inability of your eye's cones to function at moonlight levels.

Brightness Temperature. The Planck spectrum is *monotonic* with temperature at each and every frequency ν (see previous figure). Therefore a measurement of I_ν at any single frequency ν determines a temperature, called the "brightness temperature." Of course, identifying this temperature with a real physical temperature of the emitting object depends on knowing the the *rest* of the spectrum also fits a black body shape. The Planck spectrum by definition has the same brightness temperature at all frequencies.

The brightness temperature of the night sky is $\sim 200 - 300\,\mathrm{K}$ where the atmosphere is opaque, and as low as $3°\,\mathrm{K}$ (cosmic microwave background temperature) in atmospheric "windows" of transparency.



Note the "infrared windows" at $2.2\mu, 3-5\mu, 8-12\mu$, as well as the large optical and radio windows. UV and X-ray astronomy can only be done from space.

3.5.6 Radiative temperature balance of the Earth

In the most naive model, the Earth maintains that temperature at which it radiates (in the infrared) exactly the same average power as it receives from the Sun. Note that the Sun illuminates $\pi R_\oplus^2$ of projected Earth area, while the actual surface area is $4\pi R_\oplus^2$, so the average "insolation" (as it is called) is 1/4 the solar constant

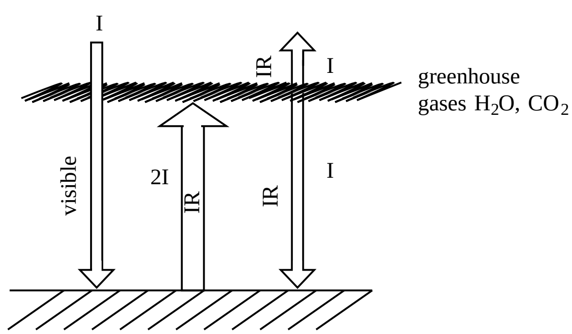
$$\overline{F}_{\rm ins} = \frac{1}{4} \times 1400\,{\rm w/m^2} \approx 3.5 \times 10^5\,{\rm erg\,s^{-1}\,cm^{-2}}\ .$$

The outgoing infrared flux at temperature T is σT^4, so:

$$\left(5.67 \times 10^{-5}\,{\rm erg\,cm^{-2}\,s^{-1}\,K^{-4}}\right) T^4 = 3.5 \times 10^5\,{\rm erg\,s^{-1}\,cm^{-2}}$$

$$\Rightarrow \qquad T = 280^\circ\,{\rm K} = 7^\circ\ {\rm C} = 44^\circ\,{\rm F} \quad \text{chilly!}$$

The reason the Earth is actually warmer than this is because of the "greenhouse effect." Atmospheric gasses absorb the IR emitted from the ground and re-radiate it to space. You might think that this is a "wash," but not so, as the following figure shows:



You see that the ground must radiate *twice* the incident flux I to stay in equilibrium. Thus,

$$\begin{aligned}(5.67 \times 10^{-5}) T_K^4 &= 7.0 \times 10^5 \\ \Rightarrow T_K &= 333^\circ\,{\rm K} = 60^\circ\,{\rm C} = 140^\circ\,{\rm F} \quad \text{hot!}\end{aligned}$$

The reason it is not this hot is that the greenhouse is not totally absorbing (note the partial "windows" between 8 and 20μ in the figure in Section 3.5.5). You can see why there are environmental worries about increasing the greenhouse gasses, however!

Keep in mind that the above calculations are idealized, because clouds reflect sunlight directly, and because transport of heat from equatorial to polar regions (by weather and ocean currents) is a big effect. It must still be true, however, that net solar flux in equals total IR flux out when averaged over time and latitude.

3.5.7 The spectral sequence of stars

Stars are classified observationally primarily by their color temperature — because, historically, this can be measured without knowing the distance. Before black body radiation was understood, an alphabetic sequence was developed empirically. This spectral sequence correlates not only strongly with color temperature, but also (not surprisingly) with the appearance of different *absorption lines*, because the ability of atoms in the outer atmospher of a star to absorb light depends on their ionization state and, thus, temperature.

The Spectral Sequence

Spectral class	Color	B-V color Index	Temperature (K)	Spectral lines	Examples
O	Blue-violet	$-.35$	$28,000 - 50,000$	Ionized atoms, especially helium	Naos (ζ Pup), Mintaka (δ Ori)
B	Blue-white	$-.16$	$10,000 - 28,000$	Neutral helium, some hydrogen	Spica (α Vir), Rigel (β Ori)
A	White	$+.13$	$7,500 - 10,000$	Strong hydrogen, some ionized metals	Sirus (α CMa), Vega (α Lyr)
F	Yellow-white	$+.42$	$6,000 - 7,500$	Hydrogen and ionized metals such as calcium and iron	Canopus (α Car), Procyon (α CMi)
G	Yellow	$+.70$	$5,000 - 6,000$	Ionized calcium and both neutral and ionized metals	Sun, Capella (α Aur)
K	Orange	$+1.2$	$3,500 - 5,000$	Neutral metals	Arcturus (α Boo), Aldebraran (α Tau)
M	Red-orange	$+1.2$	$2,500 - 3,500$	Strong titanium oxide and some neutral calcium	Antares (α Sco), Betelgeuse (α Ori)

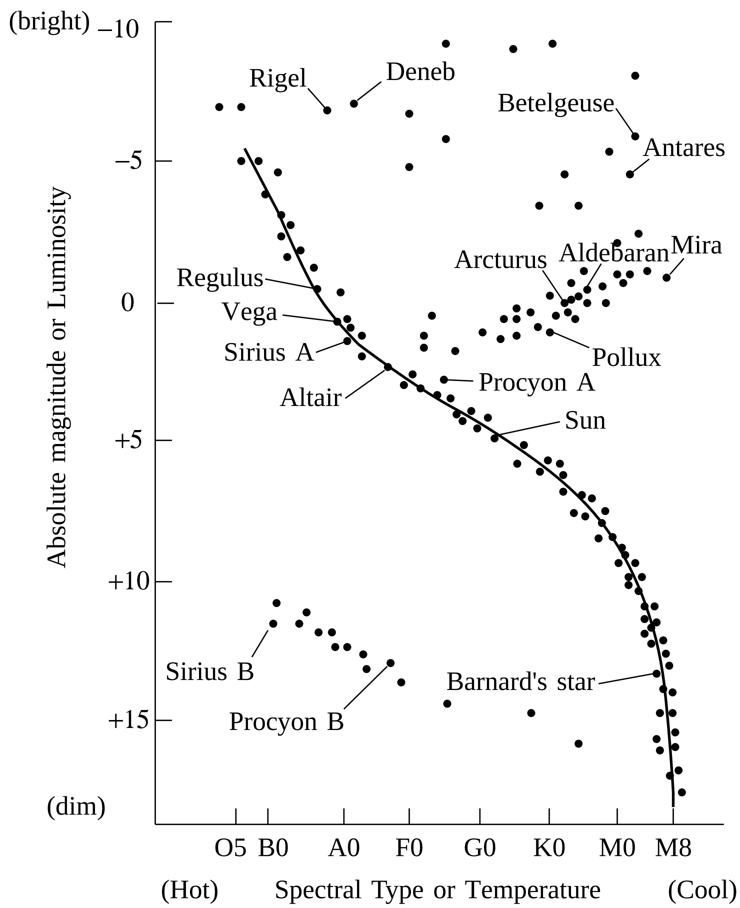
Here we are seeing the Rayleigh-Jeans continua, flux proportional to temperature, with relatively small and few absorption features. (Compare shape of curves in the figure in Section 3.5.2)

$\longleftarrow$ here we clearly see the peak of the Planck function, around 4500 Å

$\longleftarrow$ the Sun fits in here (G5)

Here we are on the Wien tail, with lots of messy absorption lines and molecular bands.

Within each spectral class there are finer steps: ... B0, B1, B2,...B9, A0, A1...A9. etc. B5 is a typical B star. The Sun is a typical G star, that is, G5. When astronomers finally knew the distances to enough stars to get their absolute magnitudes or luminosities, and plotted these against their spectral types or color temperatures, they got a big surprise:



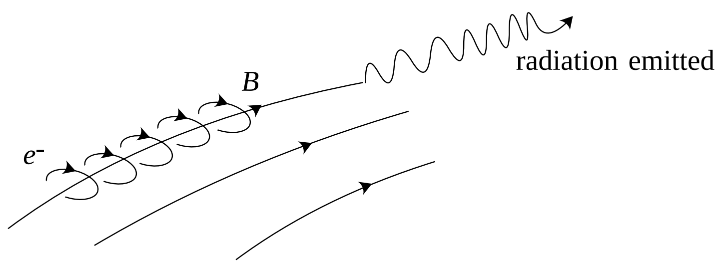
The stars are not randomly scattered, but ordered in some weird way! Uh, oh; call in the astro*physicists*! This kind of diagram is called a Hertzsprung-Russell or H-R diagram. Ejnar Hertzsprung discovered it in 1905, and Henry Morris Russell *re*-discovered it nearly 10 years later. (News travelled slow then. There were very few astronomers in the world and not many international conferences.)

3.6 Radiation emission mechanisms briefly described

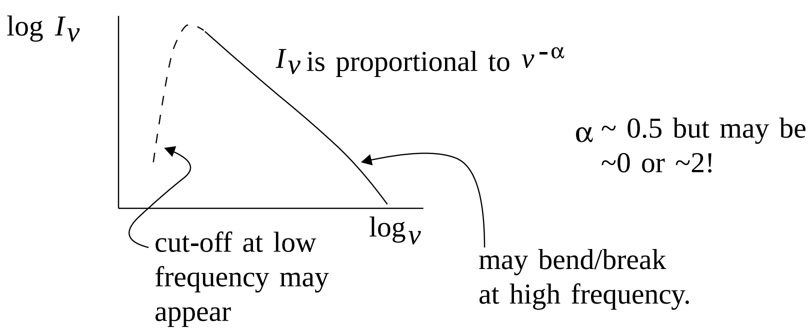
Black body radiation is radiation that has come to thermal equilibrium and lost the memory of its original "creation." Sometimes, however, we are able to see photons directly as produced by the underlying microscopic processes. These *need not* have a Planck spectrum. What, microscopically, causes the radiation we see? The *direct* cause is always the same: acceleration of a charge creates radiation. The radiation is generally classified according to what causes the acceleration. We will list here three of the most important mechanisms. (You will learn much more about this in Astronomy 150.)

3.6.1 Synchrotron radiation

Relativistic electrons being accelerated by magnetic fields (moving in helical patterns around field lines):



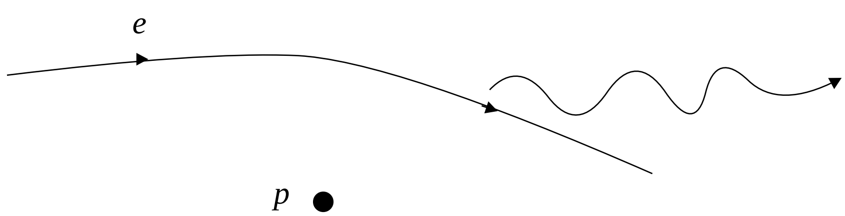
Radiation is intense; often highly polarized; usually power-law spectrum.



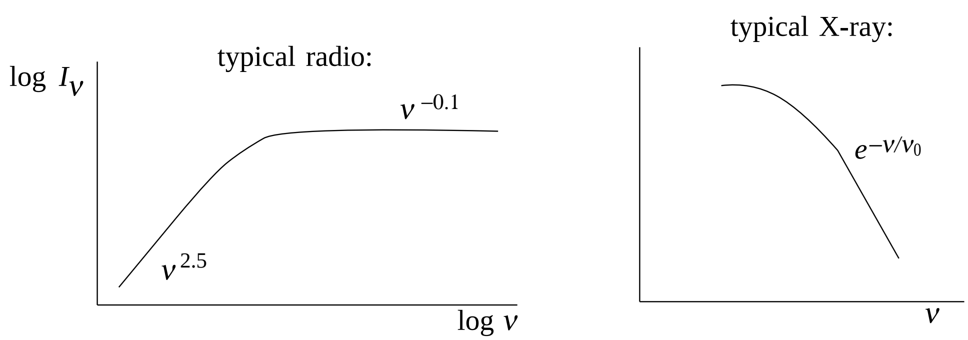
Radio emission; traces very energetic regions.

3.6.2 Thermal bremsstrahlung

Thermal electrons (i.e. electrons with Maxwell-Boltzman distributions) moving near positive ions (e.g. H nuclei). Coulomb interactions cause the acceleration.



Radiation is less intense; almost unpolarized; very flat spectrum in radio, "thermal" in X-ray (Planck-like).



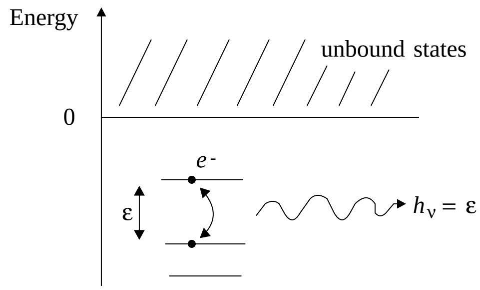
Usually associated with hot gas: X-rays from clusters of galaxies; HII regions; surface of sun.

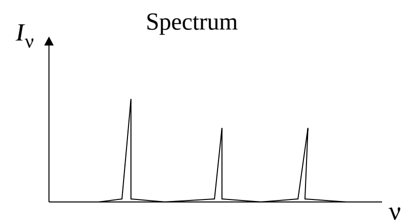
Bremsstrahlung is usually associated with hot gas: X-rays from clusters of galaxies, HII regions, or the surface of the Sun, e.g.

Synchrotron radiations and thermal bremsstrahlung are two of the common continuum emission mechanisms of gas. (Solids, e.g. surface of the Earth, are different.)

3.6.3 Line emission from atoms

Electrons jumping from higher to lower levels in atoms emit excess energy as line radiation. The distribution of the lines in frequency and intensity are characterisic of the chemical element involved.





Seen from hot gas (may see lines from many types of atom simultaneously). Molecules also make lines.

There are also *absorption* spectra, radiation absorbed by cooler material (lines or continuum), the inverse process.