4 Classical Dynamics

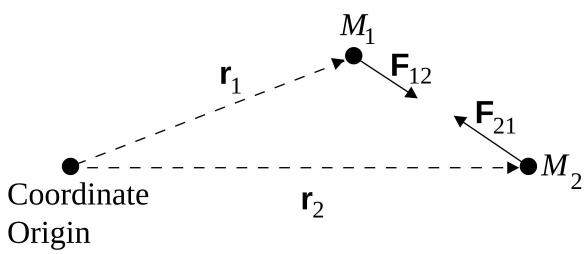
4.1 Newtonian gravity

4.1.1 Basic law of attraction

Two point masses, with mass M_1 and M_2, lying at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, attract one another. M_1 feels a force from M_2

$$\boldsymbol{F}_{12} = \text{ force on } M_1 \text{ from } M_2 \ = -\frac{GM_1M_2}{r_{12}^3}\boldsymbol{r}_{12}$$

where $\boldsymbol{r}_{12} = \boldsymbol{r}_1 - \boldsymbol{r}_2 =$ vector from 2 to 1 and $r_{12} = |\boldsymbol{r}_{12}|$



Similarly M_2 feels a force from M_1

$$\boldsymbol{F}_{21} = \text{force on } M_2 \text{ from } M_1 = -\frac{GM_2M_1}{r_{21}^3}\boldsymbol{r}_{21} = +\frac{GM_1M_2}{r_{12}^3}\boldsymbol{r}_{12} = -\boldsymbol{F}_{12}$$

where $\boldsymbol{r}_{21} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the vector from 1 to 2. We have used $\boldsymbol{r}_{21} = -\boldsymbol{r}_{12}$, and $|\boldsymbol{r}_{21}| = |\boldsymbol{r}_{12}|$. Thus, the gravitational forces are *equal* and *opposite*. This is in accordance with Newton's 3$^{\text{rd}}$ law and ensures that the composite system of $(M_1 + M_2)$ does not suddenly start moving as a whole and violating Newton's 1$^{\text{st}}$ law.

Notice also that the gravitational force at M_1 from M_2 is directed *exactly at* M_2. It is a "central" force: $\boldsymbol{F}_{12}$ is parallel to $\boldsymbol{r}_{12}$. This remark causes us to digress and mention…

4.1.2 The little known codicil to Newton's 3$^{\text{rd}}$ law

Many books, e.g. Marion, state Newton's laws in something like this form:

1. A body remains at rest or in uniform motion unless acted upon by a force.

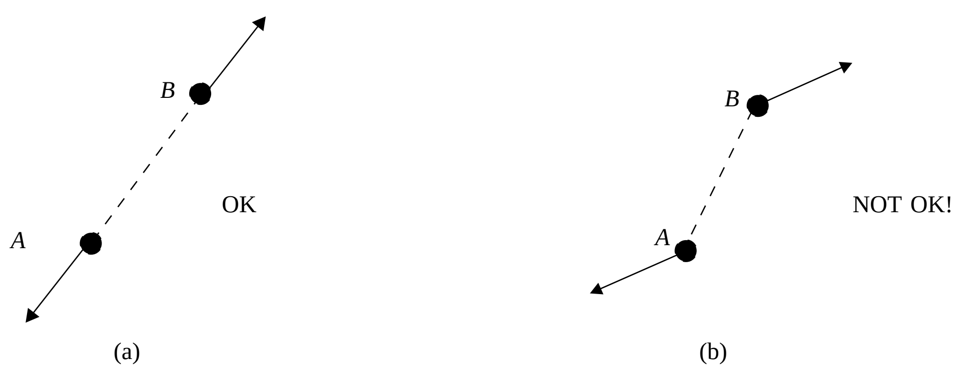
2. A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.

3. If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

This statement of 3 is WRONG! The correct statement is

3′. If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction AND ALONG THE SAME LINE.

Newton knew this, but modern text book writers have garbled it.



If (b) were possible, then we could take the two particles A and B, nail one to the hub of a wheel, the other to the center, and watch as the wheel *accelerated up to infinite angular velocity.*

Thus, while one often hears about the "central force" nature of gravitation, this is actually a property of *any* action-at-a-distance, classical force law.

You might wonder about the *magnetic* force between two moving particles in electromagnetism. This does not seem to be a central force. The explanation is that the electromagnetic field itself carries momentum and angular momentum. Classical electromagnetism cannot be written as a pure action-at-a-distance theory. That is, there is more to Maxwell's equations than just the Coulomb law.

4.1.3 Gravitational Potential

The force on M_1, due to M_2's presence can be written in terms of the gradient of a *gravitational potential:*

$$\begin{aligned}
\boldsymbol{F}_{12} &= \text{force at } M_1 \text{ due to } M_2 \\
&= \text{force at position } \boldsymbol{r}_1 \text{ due to mas at } \boldsymbol{r}_2 \\
&= -M_1 \left(\frac{\partial}{\partial x_1} \left(-\frac{GM_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \right) , \frac{\partial}{\partial y_1} \left(-\frac{GM_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \right) , \frac{\partial}{\partial z_1} \left(-\frac{GM_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \right) \right) \\
&= -M_1 \boldsymbol{\nabla}_1 V(\boldsymbol{r}_1)
\end{aligned}$$

where $\boldsymbol{\nabla}_1$ is the gradient operator at M_1; $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right)$. $V(\boldsymbol{r})$ is the gravitational potential at $\boldsymbol{r}_1$,

$$V = -\frac{GM_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} .$$

The potential at point $\boldsymbol{r}$ from a number of masses; at $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_n$ is, in general

$$V = -\sum_{i=1}^{N} \frac{GM_i}{|\boldsymbol{r} - \boldsymbol{r}_i|}$$

where the sum excludes any mass exactly at $\boldsymbol{r}$ (do not include the infinite self-energy!).

4.2 The 2-body problem

Although every mass in the universe exerts a force on every other mass, there is a useful idealization where the mutual interaction of just two masses dominates their motion. The equations of motion are then

$$\begin{aligned} M_1\ddot{\boldsymbol{r}}_1 &= -\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_1-\boldsymbol{r}_2) \\ M_2\ddot{\boldsymbol{r}}_2 &= -\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_2-\boldsymbol{r}_1)\ . \end{aligned}$$

These apply, for example, to binary stars at large separations, but small enough that the tidal effects of other stars in the galaxy can be neglected. Or, the orbit of the Earth around the Sun, to the extent that the effects of the Moon and other planets can be ignored.

4.2.1 Conservation laws for 2-body orbits

For a 2-body system, we want to solve for 6 functions of time $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$, and 12 constants will appear in the solution (equivalent to 2 vectors of initial positions and 2 vectors of initial velocities). It is most efficient to approach this problem by finding constants of the motion (to reduce the number of equations to solve) and by choosing a convenient set of coordinates in which to work.

Add the two equations of motion given above. Then, since the forces are equal and opposite, the right hand side sums to zero, and we are left with

$$M_1\ddot{\boldsymbol{r}}_1 + M_2\ddot{\boldsymbol{r}}_2 = 0\ .$$

Let the center of mass of the system be at $\boldsymbol{R}$; then by the definition of center of mass as the mass-weighted average position,

$$\boldsymbol{R} = \frac{M_1\boldsymbol{r}_1 + M_2\boldsymbol{r}_2}{(M_1+M_2)}$$

we get

$$(M_1 + M_2)\ddot{\boldsymbol{R}} = 0 \quad \text{or} \quad \ddot{\boldsymbol{R}} = 0 \ .$$

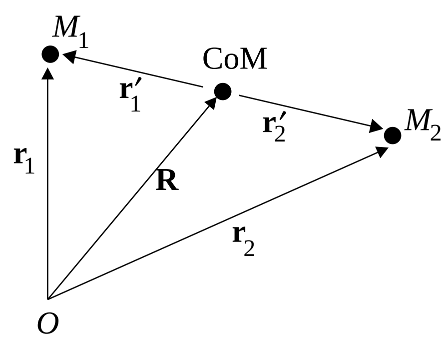
Thus, the location of the center of mass of the system is *unaccelerated*, since there is no external force on the system.

It follows that the center of mass moves at a constant velocity:

$$\begin{aligned}\dot{\boldsymbol{R}} \equiv \frac{d\boldsymbol{R}}{dt} &= \boldsymbol{v} = \text{constant} \\ \boldsymbol{R} &= \boldsymbol{R}_0 + \boldsymbol{v}_0 t \ .\end{aligned}$$

This relation expresses the "Conservation of Linear Momentum." $\boldsymbol{R}_0$ and $\boldsymbol{V}_0$ are the first 6 "constants of motion."

The next logical thing to do is to change to center-of-mass coordinates $\boldsymbol{r}'_1, \boldsymbol{r}'_2$, that is, to coordinates relative to the center of mass.



The relevant relationships are:

$$\begin{aligned}\boldsymbol{r}_1 &= \boldsymbol{R} + \boldsymbol{r}'_1 \\ \boldsymbol{r}_2 &= \boldsymbol{R} + \boldsymbol{r}'_2 \\ \boldsymbol{R} &= \boldsymbol{R}_0 + \boldsymbol{v}_0 t \ .\end{aligned}$$

Substituting these into the equations of motion, we get

$$M_1 \ddot{\boldsymbol{r}}'_1 = -\frac{G M_1 M_2}{|\boldsymbol{r}'_1 - \boldsymbol{r}'_2|^3} (\boldsymbol{r}'_1 - \boldsymbol{r}'_2)$$

$$M_2\ddot{\boldsymbol{r}}'_2 \quad = \quad -\frac{GM_1M_2}{|\boldsymbol{r}'_1-\boldsymbol{r}'_2|^3}\left(\boldsymbol{r}'_2-\boldsymbol{r}'_1\right) \ .$$

Note that the equations preserve exactly the form they had before: We could have just written down the answer without calculation. We can arbitrarily choose coordinates relative to the CoM, because the CoM frame is a good *inertial frame*. Any change of origin ($\boldsymbol{R}_0$) and/or change of velocity ($\boldsymbol{V}_0$) is allowed by (so-called) "Gallilean invariance."

So let us choose to do this: coordinates are, henceforth, measured relative to the CoM; and we drop the primes.

In CoM system,

$$\begin{aligned}
M_1\ddot{\boldsymbol{r}}_1 &= -\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_1-\boldsymbol{r}_2) \\
M_2\ddot{\boldsymbol{r}}_2 &= -\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_2-\boldsymbol{r}_1)
\end{aligned}$$

with $M_1\boldsymbol{r}_1+M_2\boldsymbol{r}_2=0$ *by definition.*

We can now construct further integrals of the motion. Consider the total angular momentum about the CoM: $\boldsymbol{L}=M_1\boldsymbol{r}_1\times\dot{\boldsymbol{r}}_1+M_2\boldsymbol{r}_2\times\dot{\boldsymbol{r}}_2$. Then

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= M_1\left(\boldsymbol{r}_1\times\ddot{\boldsymbol{r}}_1+\dot{\boldsymbol{r}}_1\times\dot{\boldsymbol{r}}_1\right)+M_2\left(\boldsymbol{r}_2\times\ddot{\boldsymbol{r}}_2+\dot{\boldsymbol{r}}_2\times\dot{\boldsymbol{r}}_2\right) \\
&= M_1\boldsymbol{r}_1\times\ddot{\boldsymbol{r}}_1+M_2\boldsymbol{r}_2\times\ddot{\boldsymbol{r}}_2 \ .
\end{aligned}$$

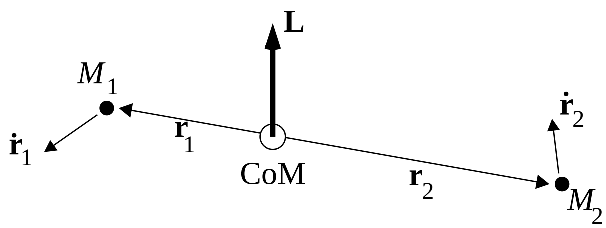
Substitute the equations of motion for $M_1\ddot{\boldsymbol{r}}_1$ and $M_2\ddot{\boldsymbol{r}}_2$, then

$$\begin{aligned}
\frac{d\boldsymbol{L}}{dt} &= -\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}\left\{\boldsymbol{r}_1\times(\boldsymbol{r}_1-\boldsymbol{r}_2)+\boldsymbol{r}_2\times(\boldsymbol{r}_2-\boldsymbol{r}_1)\right\} \\
&= \frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_1\times\boldsymbol{r}_2+\boldsymbol{r}_2\times\boldsymbol{r}_1) \\
&= 0
\end{aligned}$$

i.e. the total angular momentum of the system is conserved:

$$\boldsymbol{L}=\text{constant.}$$

That is, the angular momentum vector is fixed in direction and magnitude (because no external torques are acting).



The angular momentum provides 3 more integrals of the motion, so we are now up to 9.

We can also look at the total energy of the system:

$$\begin{aligned} E &= \text{(kinetic energy)} + \text{(potential energy)} \\ &= \left(\frac{1}{2}M_1\dot{\boldsymbol{r}}_1\cdot\dot{\boldsymbol{r}}_1 + \frac{1}{2}M_2\dot{\boldsymbol{r}}_2\cdot\dot{\boldsymbol{r}}_2\right) + \left(-\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right)\ . \end{aligned}$$

To see that this is conserved, write

$$\frac{dE}{dt} = M_1\dot{\boldsymbol{r}}_1\cdot\ddot{\boldsymbol{r}}_1 + M_2\dot{\boldsymbol{r}}_2\cdot\ddot{\boldsymbol{r}}_2 + \frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^2}\frac{d}{dt}|\boldsymbol{r}_1-\boldsymbol{r}_2|\ .$$

Now

$$\begin{aligned} \frac{d}{dt}\,|\boldsymbol{r}_1-\boldsymbol{r}_2| &= \frac{d}{dt}\,(\boldsymbol{r}_1\cdot\boldsymbol{r}_1+\boldsymbol{r}_2\cdot\boldsymbol{r}_2-2\boldsymbol{r}_1\cdot\boldsymbol{r}_2)^{1/2} \\ &= \frac{\boldsymbol{r}_1\cdot\dot{\boldsymbol{r}}_1+\boldsymbol{r}_2\cdot\dot{\boldsymbol{r}}_2-\boldsymbol{r}_1\cdot\dot{\boldsymbol{r}}_2-\dot{\boldsymbol{r}}_1\cdot\boldsymbol{r}_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \frac{(\boldsymbol{r}_1-\boldsymbol{r}_2)(\dot{\boldsymbol{r}}_1-\dot{\boldsymbol{r}}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\ . \end{aligned}$$

Therefore

$$\begin{aligned} \frac{dE}{dt} &= \dot{\boldsymbol{r}}_1\left(M_1\ddot{\boldsymbol{r}}_1+\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)+\dot{\boldsymbol{r}}_2\left(M_2\ddot{\boldsymbol{r}}_2+\frac{GM_1M_2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3}(\boldsymbol{r}_2-\boldsymbol{r}_1)\right) \\ &= 0 \end{aligned}$$

using the equations of motion. Therefore the total energy of the system,

$$E = \text{constant}$$

(because no work is being done on systems by an external force). The energy is another integral of the motion (each one equivalent in counting to one initial condition). These 10 integrals, $\boldsymbol{R}_0, \boldsymbol{v}_0, \boldsymbol{L}, E$ are general to N bodies, not just 2 (as we will see later). To *completely* specify the system we need 12 constants of integration; 10 specified: *2 remain.* These turn out to be particular to the special case of 2-bodies. That is, they are more like "initial conditions" than "conservation laws."

4.2.2 Two-body orbits

Finally we are ready to *solve* the 2-body problem. Let us recap: We have *removed* 6 integrals of the motion by working in the center of mass (= center of momentum frame). Then the other integrals of the motion are *total energy* and *total angular momentum.*

The positions of the two bodies, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, obey

$$\begin{aligned} M_1 \ddot{\boldsymbol{r}}_1 &= \frac{G M_1 M_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} (\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad (1) \\ M_2 \ddot{\boldsymbol{r}}_2 &= \frac{G M_1 M_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} (\boldsymbol{r}_2 - \boldsymbol{r}_1) \qquad (2) \end{aligned}$$

and by definition

$$M_1 \boldsymbol{r}_1 + M_2 \boldsymbol{r}_2 = 0 \ . \qquad (3)$$

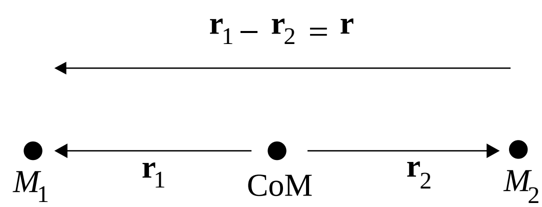
We showed that

$$\begin{aligned} M_1 \boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + M_2 \boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2 &= \boldsymbol{L} & \boldsymbol{L} &= \text{constant vector} \\ \frac{1}{2} M_1 \dot{\boldsymbol{r}}_1 \cdot \dot{\boldsymbol{r}}_1 + \frac{1}{2} M_2 \dot{\boldsymbol{r}}_2 \cdot \dot{\boldsymbol{r}}_2 - \frac{G M_1 M_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} &= E \ . & E &= \text{constant scalar} \end{aligned}$$

We now want to completely solve the system: for $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$.

It is convenient to reduce the system to simpler form by working in terms of the *separation vector* of the two bodies:

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 = \text{vector from body 2 to body 1.}$$



Then consider equation (1) divided by M_1 minus equation (2) divided by M_2:

$$\ddot{\boldsymbol{r}} = (\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \frac{G(M_1 + M_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

or

$$\ddot{\boldsymbol{r}} = -\frac{GM}{r^3}\boldsymbol{r}$$

where $M = M_1 + M_2$ is the total mass of the system. This equation says that body 1 moves as if it was of very small mass being attracted by mass $(M_1 + M_2)$ at body 2; the separation vector is accelerated by the *total mass of the system.* The fact that the 2-body problem reduces to the equation of the "one-body problem" (test mass in the central force field of a fictitious mass M) is an amazing and nonobvious result!

With the relations

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$$

$$M_1\boldsymbol{r}_1 + M_2\boldsymbol{r}_2 = 0$$

we can find how to calculate $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from the solutions for $\boldsymbol{r}$:

$$\boldsymbol{r}_1 = \frac{M_2}{M}\boldsymbol{r} \qquad \boldsymbol{r}_2 = -\frac{M_1}{M}\boldsymbol{r}\ .$$

Note that the signs are opposite (the masses must be on opposite sides of the CoM).

Eliminating $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in favor of $\boldsymbol{r}$ we find that $\boldsymbol{r}(t)$ obeys

$$\begin{aligned} \ddot{\boldsymbol{r}} &= -\frac{GM}{r^3}\boldsymbol{r} \\ \boldsymbol{r} \times \dot{\boldsymbol{r}} &= \frac{M}{M_1 M_2}\boldsymbol{L} \\ \frac{1}{2}\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} - \frac{GM}{r} &= \frac{M}{M_1 M_2}E\ . \end{aligned}$$

Notice the appearance of a quantity $\mu \equiv M_1 M_2/(M_1 + M_2)$ on the right hand sides, where we expect (one over) a mass. The quantity μ is called the *reduced mass.* Notice that μ is smaller than both M_1 and M_2.

The angular momentum equation

$$\boldsymbol{r} \times \dot{\boldsymbol{r}} = \left(\frac{M}{M_1 M_2}\right)\boldsymbol{L}$$

has an important immediate consequence. Since $\boldsymbol{r} \cdot (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = \dot{\boldsymbol{r}} \cdot (\boldsymbol{r} \times \dot{\boldsymbol{r}}) = 0$, that is, the vector cross product of two vectors is perpendicular to both of them, $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are both perpendicular to $\boldsymbol{L}$, i.e. all the motion occurs *in a plane perpendicular to* $\boldsymbol{L}$ (remember that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are proportional to $\boldsymbol{r}$, hence parallel to $\boldsymbol{r}$).

Adopting coordinates in the plane of the orbit, we can write in Cartesian coordinates:

$$\begin{aligned} \boldsymbol{L} &= (0, 0, L) \\ \boldsymbol{r} &= (x, y, 0) \\ \dot{\boldsymbol{r}} &= (\dot{x}, \dot{y}, 0)\ . \end{aligned}$$

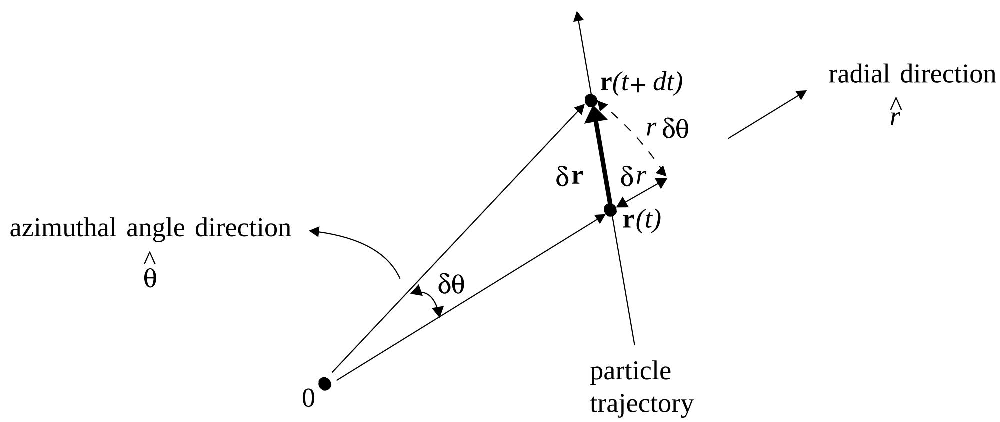
The momentum equation becomes

$$x\dot{y} - y\dot{x} = \left(\frac{M}{M_1 M_2}\right)L = \frac{L}{\mu}$$

with an energy equation

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{GM}{(x^2+y^2)^{1/2}}=\left(\frac{M}{M_1M_2}\right)E=\frac{E}{\mu}\ .$$

In principle the above two equations are the coupled differential equations for the two unknowns $x(t)$ and $y(t)$. It is simpler, however, to exploit the planar geometry in *polar* coordinates r and θ.

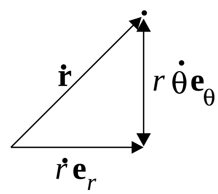


If $\boldsymbol{r}(t)\rightarrow\boldsymbol{r}(t+dt)$ in time dt, then, to first order,

$$\begin{aligned}(\text{component of }\delta\boldsymbol{r}\text{ along original radial direction}) &= \delta r\\ (\text{component of }\delta\boldsymbol{r}\text{ along direction of azimuth}) &= r\delta\theta\ .\end{aligned}$$

Therefore if, at any given time, in cylindrical coordinates (r,θ,z),

$$\begin{aligned}\boldsymbol{r} &= (r,0,0)\equiv r\boldsymbol{e}_r\\ \dot{\boldsymbol{r}} &= \frac{\boldsymbol{r}(t+\delta t)-\boldsymbol{r}(t)}{\delta t}=\left(\frac{\delta r}{\delta t},r\frac{\delta\theta}{\delta t},0\right)=\dot{r}\boldsymbol{e}_r+r\dot{\theta}\boldsymbol{e}_\theta\\ \boldsymbol{r}\times\dot{\boldsymbol{r}} &= \left(0,0,r^2\dot{\theta}\right)\\ \dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} &= \dot{r}^2+r^2\dot{\theta}^2\end{aligned}$$



"Centrifugal form of Pythagoras"

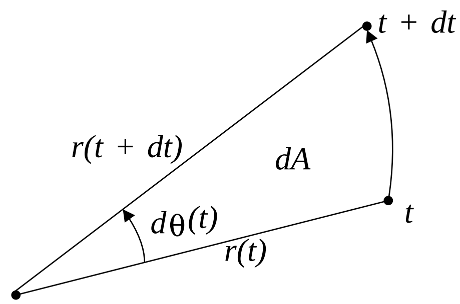
We find that the angular momentum equation is

$$r^2\dot\theta = \left(\frac{M}{M_1M_2}\right)L = \frac{L}{\mu}$$

while the energy equation is

$$\frac{1}{2}\left(\dot r^2 + r^2\dot\theta^2\right) - \frac{GM}{r} = \left(\frac{M}{M_1M_2}\right)E = \frac{E}{\mu}\ .$$

The angular momentum equation can be interpreted geometrically



In interval t to $t+dt$, the vector between the objects sweeps at area dA,

$$dA = \frac{1}{2}r\cdot r d\theta\ .$$

Therefore, the rate of change of area is

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot\theta$$

which by the first equation becomes

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{M}{M_1M_2}\right)L = \frac{L}{2\mu} = \text{constant}.$$

Thus, it is a simple consequence of the Law of momentum conservation that the rate of sweeping out of area is a constant.

This is Kepler's Second Law (1609). Kepler discovered it empirically — he had no idea what physics was behind it. Kepler was Tycho's "graduate student" (in modern terms), but Tycho would never let him look at all the data. Tycho died in 1601, and Kepler got the data then. Even today many observers would rather take their data to the grave than let theorists analyze them. (Or so it seems to theorists!)

Proceeding, we can get one equation for $r(t)$ by eliminating $\dot{\theta}$ from these equations. Then the equations can be rewritten

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= 2\frac{M}{M_1 M_2}E + \frac{2GM}{r} - \left(\frac{M}{M_1 M_2}\right)^2 \frac{L^2}{r^2} \\ \left(\frac{d\theta}{dt}\right) &= \left(\frac{M}{M_1 M_2}\right)\frac{L}{r^2} \, . \end{aligned}$$

4.2.3 Shapes of the orbits

The above equations could be solved for $r(t)$ and $\theta(t)$. The answer comes out in "elliptic integral functions." But the *shape* of the orbit can be derived by eliminating time from the equations to get an equation for $r(\theta)$:

$$\frac{dr}{d\theta} = \frac{r^2}{(L/\mu)}\left[2(E/\mu) + \frac{2(GM)}{r} - \frac{(L/\mu)^2}{r^2}\right]^{1/2}$$

where we are now using the "reduced mass"

$$\mu \equiv M_1 M_2 / M \, .$$

This is a first order differential equation that can be integrated

$$\int^r \frac{(L/\mu)\, dr}{r^2 \left[2(E/\mu) + \frac{2(GM)}{r} - \frac{(L/\mu)^2}{r^2}\right]^{1/2}} = \int^\theta d\theta = \theta - \theta_0$$

where $\theta_0 =$ constant of integration. Define a constant

$$r_0 = \frac{(L/\mu)^2}{(GM)} = \frac{L^2}{GM\mu^2}$$

which sets a *scale* for the orbit, and a second constant, ϵ

$$\epsilon^2 = 1 + \frac{2(E/\mu)(L/\mu)^2}{(GM)^2} = 1 + \frac{2EL^2}{(GM)^2\mu^3} \ .$$

Then the integral can be rewritten

$$\int^r \frac{r_0 dr}{r^2\left(\epsilon^2 - \left(1 - \frac{r_0}{r}\right)^2\right)^{1/2}} = (\theta - \theta_0)$$

which, via the substitution

$$\epsilon \cos u = 1 - \frac{r_0}{r}$$

becomes (try it!)

$$\int^u du = (\theta - \theta_0) \quad \Rightarrow u = (\theta - \theta_0) \quad \Rightarrow \frac{1}{r} = \frac{1}{r_0}\left(1 \pm \epsilon\cos(\theta - \theta_0)\right)$$

the plus-or-minus is thrown in gratuitously, because it corresponds to θ_0, an unknown constant, changing by π; so we can write the result with the more convenient + sign,

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + \epsilon\cos(\theta - \theta_0)\right) \ .$$

This is the equation of a conic section.

There are three special cases which we now consider.

Case of $\epsilon < 1$:

If $\epsilon < 1$, then $\frac{1}{r} > 0$ for all θ; so the separation between the two masses remains finite: the objects *remain bound* with

$$\frac{r_0}{1+\epsilon} \leq r \leq \frac{r_0}{1-\epsilon} \ .$$

The orbit in this case, is an *ellipse*. If we put

$$\begin{aligned} x &= r\cos(\theta-\theta_0) \\ y &= r\sin(\theta-\theta_0) \end{aligned}$$

and eliminate θ and r, the equation of the orbit becomes

$$\frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{r_0}\left(1+\frac{\epsilon x}{(x^2+y^2)^{1/2}}\right)$$

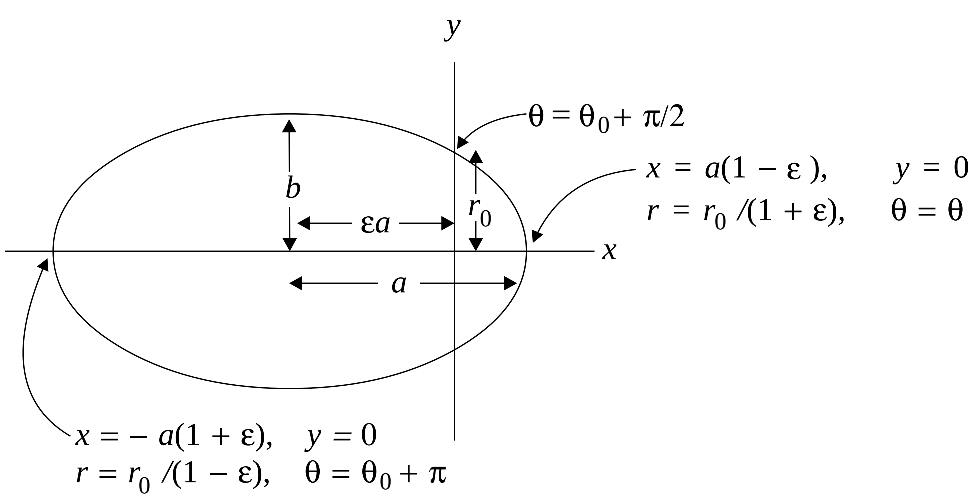
which simplifies to

$$\left(\frac{x+\epsilon a}{a}\right)^2 + \frac{y^2}{b^2} = 1$$

where

$$a = \frac{r_0}{1-\epsilon^2}\,, \quad b = \frac{r_0}{(1-\epsilon^2)^{1/2}}$$

are semi-major and semi-minor axes.



The axial ratio of the ellipse is $b/a = (1-\epsilon^2)^{1/2}$. The quantitiy r_0 is called the "semi-latus rectum." Note that $\epsilon < 1$ implies, from the definition

$$\epsilon^2 \equiv 1 + \frac{2EL^2}{(GM)^2\mu^3}$$

that $E < 0$, i.e. the *total energy of a bound system is negative.*

We can also see that the orbit is *periodic* and *closed*. Let us find the period:

$$\begin{aligned}\frac{1}{r} &= \frac{1}{r_0}\left(1+\epsilon\cos(\theta-\theta_0)\right)\\ \frac{d\theta}{dt} &= \frac{(L/\mu)}{r^2}\end{aligned}$$

are the equations governing $\theta(t)$; we can write this as a single equation for $\theta(t)$;

$$\frac{r_0^2}{(L/\mu)}\int^{\theta}\frac{d\theta}{(1+\epsilon\cos(\theta-\theta_0))^2}=\int^{t}dt\ .$$

Now, in one repeating orbit,

$$\begin{aligned}\theta \text{ increases from } \theta &\rightarrow \theta+2\pi\\ t \text{ increases from } t &\rightarrow t+\tau;,\ \tau=\text{period, so}\end{aligned}$$

$$\frac{r_0^2}{(L/\mu)}\int_0^{2\pi}\frac{d\theta}{(1+\epsilon\cos\theta)^2}=\tau\ .$$

The substitution $t=\tan\frac{1}{2}\theta$ (so $d\theta=\frac{2dt}{1+t^2}, \cos\theta=\frac{1-t^2}{1+t^2}$) renders this integral tractable. Then

$$\tau=\frac{r_0^2}{(L/\mu)}\cdot\frac{2\pi}{(1-\epsilon^2)^{3/2}}$$

Eliminating r_0 and ϵ in terms of physical variables: $(L/\mu)=(GMr_0)^{1/2}$ and $r_0=a(1-\epsilon^2)$, a the semi-major axis,

$$\tau=2\pi\frac{a^{3/2}}{(GM)^{1/2}}\ .$$

Note, incidentally (you can check) that a depends only on the total energy $a=GM_1M_2/(2E)$, while ϵ depends on both the energy and the angular momentum. For bound orbits we have now proved Kepler's laws:

1. The planets move in ellipses with the sun (the center of mass) at one focus:

$$\frac{1}{r}=\frac{1}{r_0}\left(1+\epsilon\cos(\theta-\theta_0)\right)$$

with $\epsilon < 1$ ($\epsilon = 0$ is circle).

2. A line from the Sun to a planet sweeps out equal areas in equal times:

$$\frac{dA}{dt} = \frac{1}{2}(L/\mu) \ .$$

(Note, this also works for unbound orbits.)

3. The square of the period of revolution is proportional to the cube of the semi-major axis

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM} \quad \text{or} \quad \omega^2 = \frac{GM}{a^3} \ .$$

(Note this involves the *total mass* $M = M_1 + M_2$.)

Case of $\epsilon > 1$:

Again from the definition,

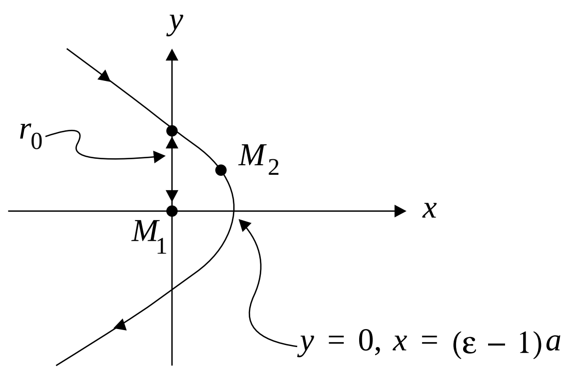
$$\epsilon = 1 + \frac{2EL^2}{(GM)^2\mu^3}$$

we see that $E > 0$, so the *total energy of system is positive.* This will imply that the orbit is unbound:

$$\frac{1}{r} = \frac{1}{r_0}(1 + \epsilon\cos(\theta - \theta_0))$$

is a hyperbola: $x = r\cos(\theta - \theta_0)$, $y = r\sin(\theta - \theta_0)$ gives, again

$$\frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{r_0}\left(1 + \frac{\epsilon x}{(x^2+y^2)^{1/2}}\right)$$
$$\Rightarrow \frac{(x-\epsilon a)^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$a = \frac{r_0}{\epsilon^2 - 1} \quad b = \frac{r_0}{(\epsilon^2-1)^{1/2}}$$
$$b = a(\epsilon^2 - 1)^{1/2}$$



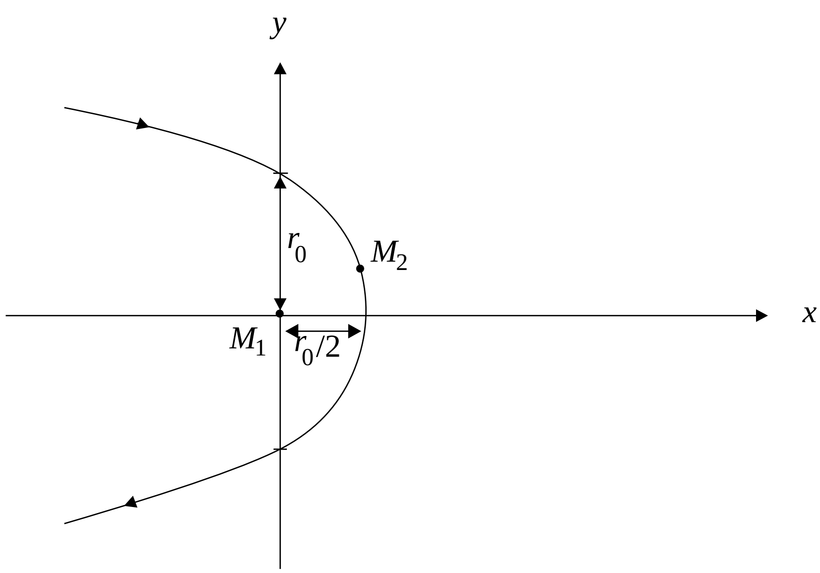
M_0 is at one focus; *Hyperbola;* single-pass orbit.

Case of $\epsilon = 1$:

From the definition of ϵ, $\epsilon = 1 \Rightarrow E = 0$, i.e. *total energy of system* is zero. In this case the orbit is a *parabola*:

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + \cos(\theta - \theta_0)\right)$$

$$\Rightarrow \qquad y^2 = r_0^2 - 2r_0 x$$



Parabola. Single pass orbit.

4.2.4 Orbital elements

From an observational point of view, a *complete* description of a Keplerian orbit (e.g., of a planet or asteroid) is one that establishes its size, shape, and orientation in 3 dimensional space, and also gives enought information to determine where along the orbit the object is at any given time. One such complete description consists of 6 "orbital elements." (There must be 6, because the complete description is equivalent to giving a vector of positions and a vector of velocities at some fiducial $t = 0$.)

In the following picture, the actual orbit is the inner ellipse, while the "tilted circles" are just to show (separately) the orbital plane and ecliptic plane (equator of the coordinate system used). The symbol Υ is the "first point of Aries," the origin of longitude in these coordinates. Do not be fooled by an optical illusion: The orbit is in the orbital plane and is an ellipse — not a circle — whose major axis is $\overline{A'SA}$. The perihelion (closest approach of the orbit to the sun) is A, while P is the position of the planet "right now."

The 6 orbital elements are defined as [2 "shape" parameters]:

semimajor axis	a	half the length of $\overline{AA'}$
eccentricity	e	$(1 - b^2/a^2)^{1/2}$ if b is the semiminor axis

[2 parameters define the orbital plane:]

inclination	i	angle ($\angle QNB$) between the orbital plane and the ecliptic plane
longitude of the ascending node	Ω	angle ($\angle \Upsilon SN$) from first point of Aries Υ to the line of nodes $N'N$. "Ascending" means pick the node where the planet crosses the ecliptic from south (below) to north (above.)

[1 parameter orients the orbit in its plane:]

argument of perihelion	ω	angle ($\angle NSA_1$) from the ascending node, measured in the plane of the orbit, to the perihelion point A.

[1 parameter specifies orbital phase:]

time of perihelion passage	T	one of the precise times that the object passes through to the point A

Note that P, the period is not an orbital element, since you can compute it from a by

$$P \text{ (in years) } = [a \text{ (in AU) }]^{3/2} .$$

Instead of ω, people sometimes quote the "longitude of perihelion"

$$\varpi \equiv \Omega + \omega = \angle \Upsilon SN + \angle NSA_1 .$$

(The symbol ϖ is actually a weird form of "pi," but most astronomers call it "pomega"!) Note that the two summed angles are measured along two different planes! This is not *really* the longitude of *anything*, but it approximates it when the inclination i is small.

Also, instead of T, people sometimes give the "longitude" L of the planet at a specified epoch (time). Here the longitude is again a kind of phoney:

$$L \equiv \angle \Upsilon N + \angle NP_1 \, .$$

In case you want to compute locations of the planets, here are the values of their orbital elements, called "ephimerides" (the plural of "ephemeris"):

Planets: Mean Elements

(For epoch J2000.0 = JD24515450 = 2000 January 1.5)

Planet	**Inclination (i)**	**Eccentricity (e)**
Mercury	7°00′17.″95051	0.2056317524914
Venus	3°23′40.″07828	0.0067718819142
Earth	0.0	0.0167086171540
Mars	1°50′59.″01532	0.0934006199474
Jupiter	1°18′11.″77079	0.0484948512199
Saturn	2°29′19.″96115	0.0555086217172
Uranus	0°46′23.″50621	0.0462958985125
Neptune	1°46′11.″82795	0.0089880948652
Pluto	17°08′31.″8	0.249050

	Mean Longitude of Node (Ω)	**Mean Longitude of Perihelion (ϖ)**	**Mean Longitude of Epoch (L)**
Mercury	48°19′51.″21495	77°27′22.″02855	252°15′03.″25985
Venus	76°40′47.″71268	131°33′49.″34607	181°58′47.″28304
Earth	0.0	102°56′14.″45310	100°27′59.″21464
Mars	49°33′29.″13554	336°03′36.″84233	355°25′59.″78866
Jupiter	100°27′51.″98631	14°19′52.″71326	34°21′05.″34211
Saturn	113°39′55.″88533	93°03′24.″43421	50°04′38.″89695
Uranus	74°00′21.″41002	173°00′18.″57320	314°03′18.″01840
Neptune	131°47′02.″60528	48°07′25.″28581	304°20′55.″19574
Pluto	110°17′49.″7	224°08′05.″5	238°44′38.″2

	Mean Distance (AU)	**Mean Distance (10^{11} m)**
Mercury	0.3870983098	0.579090830
Venus	0.7233298200	1.08208601
Earth	1.0000010178	1.49598023
Mars	1.5236793419	2.27939186
Jupiter	5.2026031913	7.78298361
Saturn	9.5549095957	14.29394133
Uranus	19.2184460618	28.75038615
Neptune	30.1103868694	45.04449769
Pluto	39.544674	59.157990

4.2.5 The mass of the Sun and the masses of binary stars

We have seen that the sidereal period of a planet, τ, is related to the semi-major axis of its orbit, a, by

$$\tau^2 = \left(\frac{4\pi^2}{GM}\right) a^3 ,$$

where

$$M = M_\odot + m_{\rm planet} .$$

This formula has been applied to calculate GM on the last column of the following table.

	Observed		Inferred
Planet	Sidereal period (days)	Semi-major axis of orbit (AU)	GM (10^{26} cm^3 s^{-1})
Mercury	87.969	0.387099	1.32714
Venus	224.701	0.723332	1.32713
Earth	365.256	1.000000	1.32713
Mars	686.980	1.523691	1.32712
Jupiter	4332.589	5.202803	1.32839
Saturn	10759.22	9.53884	1.32750
Uranus	30685.4	19.1819	1.32715
Neptune	60189	30.0578	1.32723
Pluto	90465	39.44	1.32727

Therefore, from the low-mass planets,

$$GM_\odot = 1.32713 \times 10^{26}\,\mathrm{cm}^3\,\mathrm{s}^{-1}\ .$$

This is known very accurately, to ~ 1 part in 10^6. Amazingly, G itself, from laboratory experiments, is only known to about 1 part in 10^4 as $(6.670 \pm 0.004) \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{s}^{-1}\,\mathrm{g}^{-1}$. So our knowledge of the mass of the Sun is limited by laboratory physics, not astronomical observations!

$$M_\odot = (1.989 \pm 0.001) \times 10^{33}\,\mathrm{gm}\ .$$

We can also estimate the mass of Jupiter, M_J:

$$\begin{aligned} G(M_\odot + M_J) &= 1.32839 \times 10^{26}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \\ G(M_\odot + M_E) &= 1.32713 \times 10^{26}\,\mathrm{cm}^3\,\mathrm{s}^{-1}\ , \end{aligned}$$

where M_E is the mass of the Earth. Jupiter has about an 0.1% effect on GM. Thus $M_J \sim 0.1\%$ of solar mass. Furthermore,

$$G(M_J - M_E) = 1.26 \times 10^{23}\,\mathrm{cm}^3\,\mathrm{s}^{-1}$$
$$\frac{M_J - M_E}{M_\odot} = 0.000949\ ,$$

which gives us

$$M_J \approx 0.000949 M_\odot \simeq 1.89 \times 10^{30}\,\mathrm{g}\ .$$

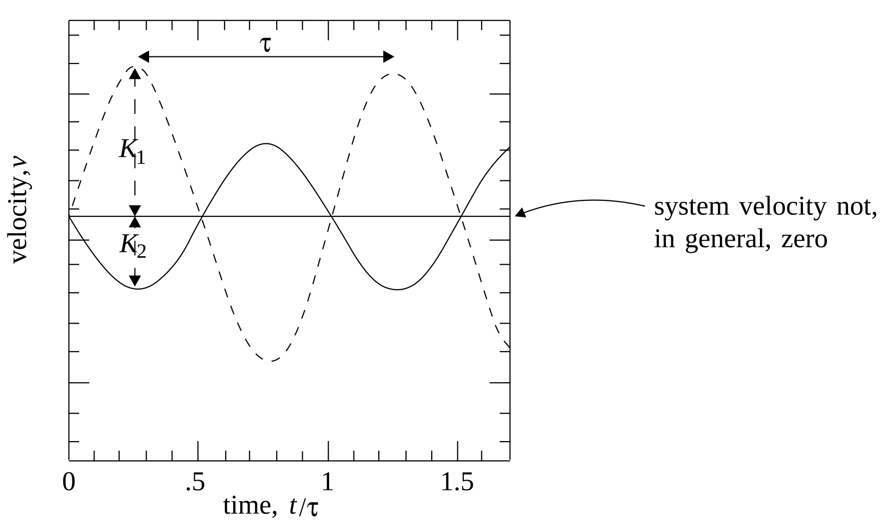
This is fairly inaccurate (1 part in 10^3), and it is a good illustration of an important principle: normally, we measure planet masses by their effects on the *satellites* or one another. Artificial satellites (e.g., Voyager II at Uranus) are also good!

This works for our Sun, but what about other stars? Their planets are too faint: there are no low-mass test particles that we can see around these stars. But $\gtrsim 50\%$ of all stars are in multiple star systems. So we can use the *relative motions of binary stars* to get stellar masses.

The periods of binary stars vary typically from a few hours (e.g., 6^h for Mirzam, alias β Ursa Majoris) to hundreds of years (e.g., 700 yr for 61 Cygni). Some are even shorter, but for these it is likely that the stars are highly distorted by tidal effects and their masses will be subject to systematic errors (e.g., "semidetached" or "contact" binaries).

A variety of data *may* be available for a binary star. We may see the motion of one star around the other on the sky (which is rare), or we may see only variations in the velocities of the stars (as seen from Doppler shifts in their spectra — this is more common). Binary stars may be single-lined (see light from only one star) or double-lined.

Consider, for example, the second case, a "double-lined spectroscopic binary star." Here, we see the two stars' spectra superimposed, and we can measure the radial velocities for both stars, but the stars are too close together for us to see their orbits by measuring the changing positions of the stars relative to one another. Then what we see are star velocities like



Immediately, we see that we have three observables:

- the orbital period, τ
- the peak velocity of the less massive star 1, K_1 (the more massive star moves more slowly)
- the peak velocity of star 2, K_2

The shape of the curves is here sinusoidal, which implies circular orbits, $\varepsilon = 0$. Other shapes are possible for $O \leq \varepsilon \leq 1$.

What can we get from these data? To simplify things, suppose the stars are in a circular orbit, $\varepsilon = 0$ (not necessary — we could fit ε from the velocity data if we wanted to, but this makes things harder). Then our orbit equations are

$$M_1 r_1 + M_2 r_2 = 0 \qquad \text{center of mass at rest}$$
$$\frac{d\theta}{dt} = \frac{L/\mu}{r_0^2} = \text{constant} = \frac{2\pi}{\tau} = \Omega \qquad \text{constant angular velocity } \Omega\text{; circular orbit.}$$

But the period is related to the separation of the stars, r (the same as the semimajor axis, for a circular orbit), by

$$\tau^2 = \frac{4\pi^2 r^3}{GM} \qquad \text{for circular orbits,}$$

and the speeds of the stars in their orbits are $v_1 = \Omega r_1$ and $v_2 = \Omega r_2$. If the orbital plane is parallel to the line of sight, the peak velocities we would see for the stars are Ωr_1 and Ωr_2. In general, however, this will not be the case (see figure on the next page).

At inclination angle i (i = angle between line of sight and normal to orbital plane)

$$\begin{aligned} K_1 &= \Omega r_1 \sin i = \frac{2\pi}{\tau} r_1 \sin i \\ K_2 &= \Omega r_2 \sin i = \frac{2\pi}{\tau} r_2 \sin i \ . \end{aligned}$$

If $i = 0$, we see no velocity. Thus, the separation of the stars is

$$r = r_1 + r_2 = \frac{\tau(K_1 + K_2)}{2\pi \sin i} \ .$$

Using the previous formula for τ^2 and eliminating r, the period equation then gives a total mass

$$M_1 + M_2 = \frac{\tau}{2\pi G} \cdot \frac{(K_1 + K_2)^3}{\sin^3 i} \ .$$

The center of the mass provides the constraint $M_1 r_1 = M_2 r_2$, so the mass ratio is

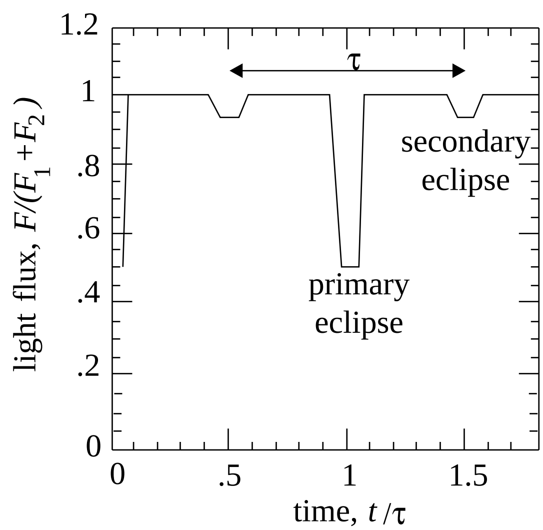
$$\frac{M_2}{M_1} = \frac{r_1}{r_2} = \frac{K_1}{K_2} \ .$$

Putting these together,

$$\begin{aligned} M_1 \sin^3 i &= \frac{1}{2\pi G} \tau (K_1 + K_2)^2 K_2 \\ M_2 \sin^3 i &= \frac{1}{2\pi G} \tau (K_1 + K_2)^2 K_1 \ . \end{aligned}$$

Here the right hand sides are completely known, so we infer the left-hand sides. These data only give masses if we know the inclination of the orbit, i. But, in general, we will not know this, so our data only measure a combination of masses and i.

For one special group of stars, we do get M_1 and M_2 directly — from "eclipsing" binaries, where the light curve shows an eclipse (see figure).



Here, the stars are eclipsing one another . This scenario is only possible if $i \sim 90°$ (provided that the stellar radii $\ll$ separation, which we can check in the above equations; if this is not true, the stars are so distorted that we cannot use this method anyway). Thus, $\sin i \sim 1$. Errors in i have little effect on the masses derived, because $\sin i$ changes slowly at $i \sim 90°$.

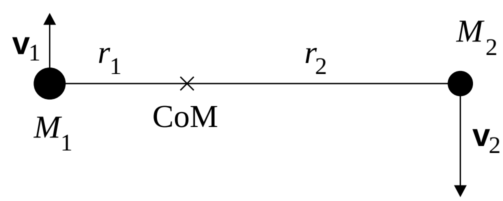
In fact, the best data on masses of stars comes from eclipsing spectroscopic double-lined binaries, even though these are hard to find.

An alternative approach is to measure many noneclipsing binaries, assume i to be selected at random, and use statistics to derive the distribution of masses, M_1 and M_2.

4.2.6 Supernovae in binary systems

It is quite common for a supernova to be in a binary system. In fact, some supernovae are *caused* by a companion star's dumping mass on a star, until the latter explodes (section 4.3.4, below). What happens to the binary system when the explosion occurs? It turns out that the force of the explosion is not an important effect on the star, but its *mass loss* is.

Before the supernova we have (say) two stars in circular orbit:



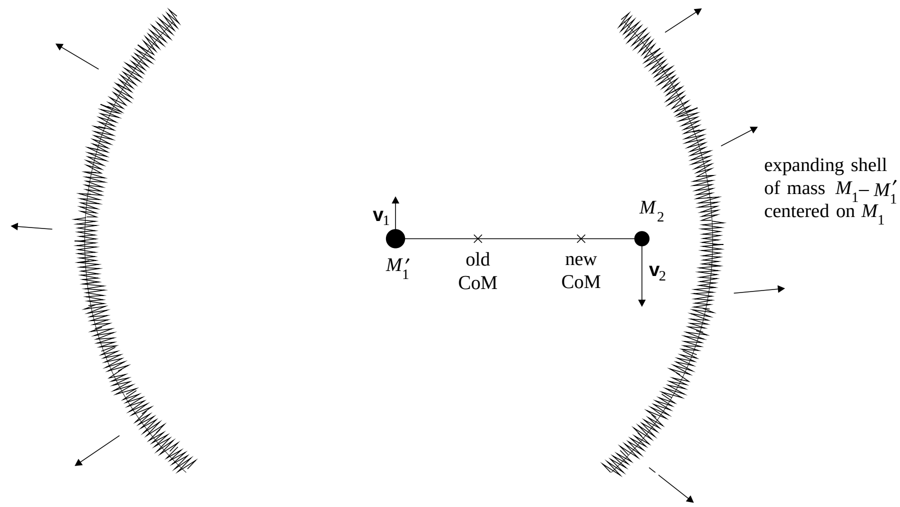
Putting the CoM at rest, we have

$$\begin{aligned} M_1 \boldsymbol{v}_1 + M_2 \boldsymbol{v}_2 &= 0 \\ M_1 \boldsymbol{r}_1 + M_2 \boldsymbol{r}_2 &= 0\ . \end{aligned}$$

Let M_1 be the mass that explodes. Typically it is the more massive star that explodes, so $M_1 > M_2$. When it explodes, it quickly (and spherically, say) blows off most of its mass, so that its new mass M_1' is

$$M_1' = M_1 - \Delta M$$

and we have



A first important effect is that the remaining binary is *not at rest.* Let's calculate its new CoM velocity v_c:

$$M_1'\boldsymbol{v}_1 + M_2\boldsymbol{v}_2 = (M_1' + M_2)\boldsymbol{v}_c \ .$$

Or, using $\boldsymbol{v}_1 = -M_2\boldsymbol{v}_2/M_1$ and $M_1' = M_1 - \Delta M$,

$$(M_1 - \Delta M)\left(-\frac{M_2}{M_1}\boldsymbol{v}_2\right) + M_2\boldsymbol{v}_2 = (M_1 - \Delta M + M_2)\boldsymbol{v}_c$$

giving

$$\boldsymbol{v}_c = \frac{\Delta M M_2}{M_1(M_1 + M_2 - \Delta M)}\boldsymbol{v}_2 \ .$$

Notice that as $\Delta M \to M_1$ (all-star explosion!), $\boldsymbol{v}_c \to \boldsymbol{v}_2$, as it must.

Typical values might be $M_1 = 10M_\odot$, $M_2 = 5M_\odot$, $\Delta M = 8.5M_\odot$, leaving a $1.5M_\odot$ neutron star for M_1', so

$$\boldsymbol{v}_c = \frac{8.5 \times 5}{10 \times 6.5}\boldsymbol{v}_2 = 0.65\boldsymbol{v}_2 \ .$$

For close binaries, v_2 can be hundreds of kilometers per second, so the system really takes off!

In fact, we need to check whether the binary remains bound at all! The easiest way to do this is to see if its total energy in the new CoM is positive or negative just after the explosion:

$$\begin{aligned} E' &= \begin{pmatrix} \text{total energy} \\ \text{in new CoM} \end{pmatrix} = \begin{pmatrix} \text{total} \\ \text{kinetic} \\ \text{energy} \end{pmatrix} - \begin{pmatrix} \text{kinetic} \\ \text{energy} \\ \text{of CoM} \end{pmatrix} + \begin{pmatrix} \text{potential} \\ \text{energy} \end{pmatrix} \\ &= \left(\frac{1}{2}M_1' v_1^2 + \frac{1}{2}M_2 v_2^2\right) - \frac{1}{2}\left(M_1' + M_2\right) v_c^2 - \frac{G M_1' M_2}{r} \,. \end{aligned}$$

Recall that (Section 4.2.2) the separation vector $\boldsymbol{r}$ and the relative velocity $\boldsymbol{v}_2 - \boldsymbol{v}_1$ satisfy Kepler's Laws with the total mass. So for circular orbits,

$$\frac{G(M_1 + M_2)}{r} = (\boldsymbol{v}_2 - \boldsymbol{v}_1)^2 \,.$$

Substituting this, and $\boldsymbol{v}_c$ previously obtained, and also eliminating M_1' in favor of $M_1 - \Delta M$ and v_1 in favor of $v_2 M_2/M_1$, we get

$$\begin{aligned} E' &= \frac{1}{2}(M_1 - \Delta M)\left(\frac{M_2 v_2}{M_1}\right)^2 + \frac{1}{2}M_2 v_2^2 - \frac{1}{2}(M_1 + M_2 - \Delta M) \\ &\times \left[\frac{\Delta M M_2 v_2}{M_1(M_1 + M_2 - \Delta M)}\right]^2 - \frac{(M_1 - \Delta M)M_2}{M_1 + M_2}\left[\left(1 + \frac{M_2}{M_1}\right) v_2\right]^2 \,. \end{aligned}$$

Now a fairly amazing algebraic simplification (I used the "`Simplify [ ]`" function in Mathematica on the computer) gives

$$E' = -\left(\frac{1}{2}M_2 v_2^2\right) \frac{(M_1 - \Delta M)(M_1 + M_2)}{M_1^2(M_1 + M_2 - \Delta M)}(M_1 + M_2 - 2\Delta M) \,.$$

Notice that all the terms are positive (since $\Delta M < M_1$) *except* the last one. We see that a condition for E to be positive — unbinding the binary — is

$$\Delta M > \frac{1}{2}(M_1 + M_2)$$

that is, loss of more than exactly *half* the total mass of the system. In the previous numerical example we have

$$8.5 > \frac{1}{2}(10+5)$$

so the neutron star is unbound and flies away at high velocity. In fact, pulsars (which are neutron stars) are often seen to be leaving the galactic plane, where they are formed, at high velocity for just this reason.

You might wonder if there isn't a simpler way to derive the above "half of total mass" result. There *is* if you are comfortable with expressing the energy of a 2-body system in terms of the reduced mass μ (as in above section 4.2.2). Then (as Mike Lecar pointed out to me) you can write the total energy of a *moving or stationary* binary system as

$$E = \frac{1}{2}Mv_{\rm CoM}^2 + \mu\left(\frac{1}{2}v^2 - \frac{GM}{r}\right)$$

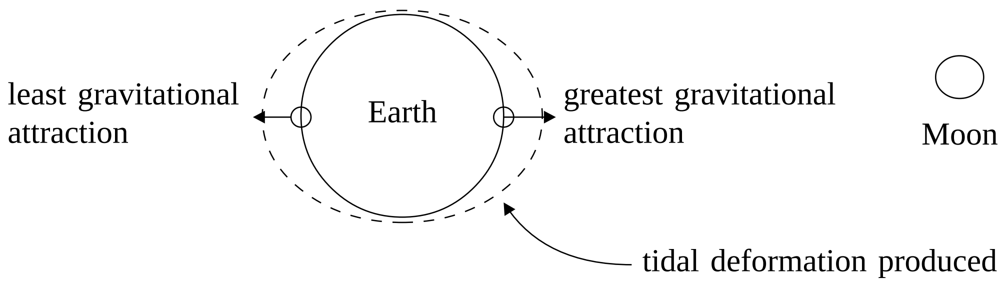
where M is total mass, $v_{\rm CoM}$ is the center of mass velocity, and v is the relative velocity (that is, $|\dot{\boldsymbol{r}}|$). The first term has no effect on whether the binary is bound or unbound. Rather, it is just a question of whether the second term is positive or negative. For an initial circular orbit, we have

$$v^2 = \frac{GM}{r} \quad \text{so that} \quad \left(\frac{1}{2}v^2 - \frac{GM}{r}\right) = -\frac{1}{2}\frac{GM}{r}\ .$$

After the supernova, v^2 is (instantaneously) the same, while μ and M change. You can see that losing more than half of M causes $v^2/2 - GM/r$ to change sign (become unbound).

4.3 Tides and Roche effects

When two bodies are in orbit around each other, the otherwise spherical gravitational field around each body is distorted by the gravitational attraction of the other body. For the Earth-Moon system (e.g.) this causes the surface of the oceans (and to some extent the solid Earth as well) to be deformed into the familiar tidal shape:



Our goal is to understand the above picture *quantitatively*.

The surface of the ocean is an *equipotential*, because if it were not — that is, if there were a potential gradient along the surface — there would be a force causing the water to flow "downhill" until it filled up the potential "valley."

But is it a purely gravitational potential we must consider? No, because the "stationary frame" in which things can come to equilibrium is the *rotating frame* in which the Earth and Moon are fixed. (We are here assuming a circular orbit. Things would be more complicated if the orbit were significantly eccentric.) So there is also a centrifugal force and corresponding centrifugal potential. If the orbital angular velocity is $\boldsymbol{\omega}$, with

$$\omega^2 = \frac{G(M_1 + M_2)}{R^3} ,$$

then the centrifugal acceleration at a position $\boldsymbol{r}$ with respect to the CoM is $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$, so the centrifugal potential (whose negative gradient is the acceleration) is $\frac{1}{2}|\boldsymbol{\omega} \times \boldsymbol{r}|^2$ or, in the orbital plane, $\frac{1}{2}\omega^2 r^2$. The total potential in the rotating frame is thus

$$\Phi = -\frac{GM_1}{|\boldsymbol{r} - \boldsymbol{r}_1|} - \frac{GM_2}{|\boldsymbol{r} - \boldsymbol{r}_2|} - \frac{1}{2}\omega^2 r^2 .$$

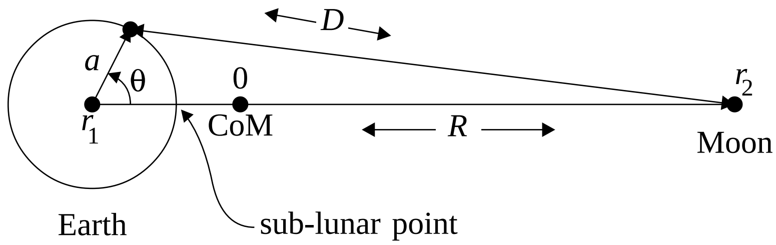
The figure shows an example of such a potential. Notice that the downhill gradient tries to make a test particle either fall into one of the potential wells of the masses *or* be flung off to infinity by centrifugal force. We will come back to this general case below in Section 4.3.4.

4.3.1 Weak tides

The Moon (M_2) is far from the Earth (M_1) and can be regarded as a point mass. Since the size of the Earth is likewise small compared to the Earth-Moon distance R, we can write the Law of Cosines

$$D^2 = R^2 + r^2 - 2Rr\cos\theta$$

and then expand in the small ratio r/R.

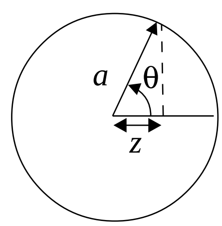


$$-\frac{GM_2}{|\boldsymbol{r}-\boldsymbol{r}_2|} = -\frac{GM_2}{D} = \frac{-GM_2}{(R^2+a^2-2Ra\cos\theta)^{1/2}}$$
$$= -\frac{GM_2}{R}\left(1+\frac{a^2}{R^2}-2\frac{a}{R}\cos\theta\right)^{-1/2} .$$

We need to do the binomial expansion to *second* order to consistently pick up all terms of order a^2/R^2. This gives

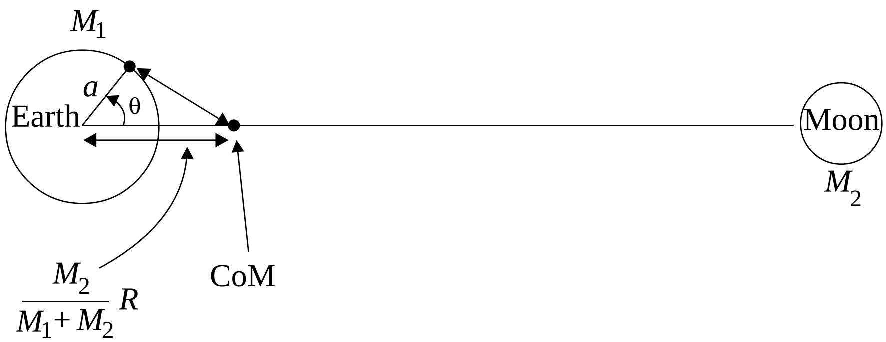
$$= -\frac{GM_2}{R}\left[1+\frac{a}{R}\cos\theta+\frac{1}{2}(3\cos^2\theta-1)\frac{a^2}{R^2}+O\left(\frac{a}{R}\right)^3\right] .$$

(If you know about multipoles, you will see the Legendre polynomials in $\cos\theta$ lurking here.) Note that the term $(a/R)\cos\theta$ actually varies *linearly* in the z direction from M_1 to M_2, since $z = a\cos\theta$, so the gradient of this part of the potential is a constant force.



Now, for the centrifugal potential term, we again apply the law of cosines,

$$-\frac{1}{2}\omega^2 r^2 = -\frac{1}{2}\omega^2\left[\left(\frac{M_2}{M_1+M_2}R\right)^2+a^2-2\left(\frac{M_2}{M_1+M_2}R\right)a\cos\theta\right] .$$



So, collecting the three terms in the overall potential Φ, and substituting for ω^2 (formula in 4.3),

$$\begin{aligned}\Phi \;&=\; -\frac{GM_1}{a} - \frac{GM_2}{R}\left[1 + \frac{a}{R}\cos\theta + \frac{1}{2}\left(3\cos^2\theta - 1\right)\frac{a^2}{R^2}\right] \\ &\quad -\frac{1}{2}\frac{G(M_1+M_2)}{R^3}\left[\left(\frac{M_2}{M_1+M_2}\right)^2 R^2 + a^2 - 2\left(\frac{M_2}{M_1+M_2}\right)Ra\cos\theta\right] .\end{aligned}$$

Look carefully and you will see that the term in $(a/R)\cos\theta$ exactly cancels out. This is not coincidence: it is because the constant force term is exactly canceled by the centrifugal force that keeps the bodies in a circular orbit. Also note that there are terms with no a or θ dependence: Since these are just constants, they produce no gradients and can be ignored. So we get, combining the remaining terms,

$$\Phi(a,\theta) = -\frac{GM_1}{a} - \frac{1}{2}\frac{Ga^2}{R^3}\left(3M_2\cos^2\theta + M_1\right) + \text{constant}.$$

This is the local tidal potential near the Earth. To get a better feeling for its shape, let us expand a in terms of height h above the radius of the Earth ("mean sea level"):

$$a = R_\oplus + h\ , \quad a^2 = R_\oplus^2\left(1 + \frac{2h}{R_\oplus} + \cdots\right)\ , \quad a^{-1} = R_\oplus^{-1}\left(1 - \frac{h}{R_\oplus} + \cdots\right)\ .$$

Here we have used the binomial expansion to get a^2 and a^{-1} to first order in $h/R_\oplus$. Now, to this order,

$$\Phi(h,\theta) = -\frac{GM_1}{R_\oplus}\left(1-\frac{h}{R_\oplus}\right) - \frac{1}{2}\frac{GR_\oplus^2}{R^3}\left(1+\frac{2h}{R_\oplus}\right)(3M_2\cos^2\theta + M_1) + \text{constant}.$$

We can now absorb into the constant all terms that depend on neither h nor θ. Doing this, and with some rearranging of terms (putting all multipliers of $\cos\theta$ together, and all remaining multipliers of h together), we get

$$\begin{aligned}\Phi(h,\theta) &= \frac{GM_1}{R_\oplus^2}\left[h\left(1-\frac{R_\oplus^4}{R^4}\right) - \cos^2\theta\frac{M_2}{M_1}\frac{R_\oplus^4}{R^3}\left(\frac{3}{2}+\frac{h}{R}\right)\right] + \text{constant} \\ &\approx g\left[h - \frac{3}{2}\left(\frac{M_2}{M_1}\right)\left(\frac{R_\oplus}{R}\right)^3 R_\oplus\cos^2\theta\right] + \text{constant}.\end{aligned}$$

Here the approximate equality means we are neglecting terms that are smaller than the dominant terms by either factors of $R_\oplus/R$ or h/R (both being assumed small). Note that g is the acceleration of gravity at the surface of the Earth.

The surface of the ocean (in the figure at the beginning of 4.2) is an equipotential, so it must have $\Phi(h,\theta) = \text{constant}$, implying

$$h = \frac{3}{2}\left(\frac{M_2}{M_1}\right)\left(\frac{R_\oplus}{R}\right)^3 R_\oplus\cos^2\theta + \text{constant}.$$

Since $\cos^2\theta$ varies between 1 ("front and back" of Earth) and zero ("sides" of Earth) the other factors give the height of the tide (high minus low). Putting in $M_2/M_1 \approx 1/81$, $R_\oplus \approx 6400\,\text{km}$, $R \approx 380000\,\text{km}$, we get $h = 54\,\text{cm}$.

The Sun also generates a significant tide that, to lowest order, just adds with that of the moon (but with a different origin for the direction θ, pointing to the Sun). For the Sun

$$h = \frac{3}{2}\times(332000)\times\left(\frac{6400\,\text{km}}{1.5\times10^8\,\text{km}}\right)^3\times 6400\,\text{km} = 25\,\text{cm}.$$

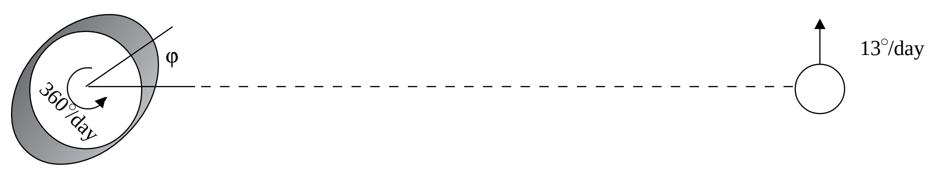
Notice that we never assumed that M_1/M_2 was either $\ll 1$ or $\gg 1$, so our formulas apply both for the Sun and for the Moon. It is a numerical coincidence that the tides they raise are of the same order (though some have speculated that the resulting more-variable tides produced could have been somehow necessary to mix the oceans in a way that furthered the evolution of life — or its emergence from the sea).

4.3.2 Tidal drag and the lengthening of the day

Real tides are not exactly the size we have calculated, primarily because the Earth is rotating. Thus the continents are "dragged through" the oceanic tidal bulges, causing water to pile up on the continental shelves, flow into and out of bays, etc. There can also be resonance effects in some cases like the famous Bay of Fundy.

This kind of "friction" between the rotating Earth and the tidal bulges produces two effects.

1. The tidal bulge is, on average, dragged ahead of where it would be if it "pointed at" the Moon.



2. The drag slows down the rotation of the Earth.

Since angular momentum must be conserved, as the Earth *loses* angular momentum, it must be *gained* by the Moon's orbital motion! How does this come about? The non-symmetric bulge (angle φ in above picture) causes a gravitational torque back on the Moon which transfers exactly the right amount of angular momentum. This *must* work out exactly because angular momentum is truly conserved! (Note that energy is not here conserved because friction is present between the ocean and the rotating Earth.)

Increasing the Moon's orbital angular momentum causes its orbit to gradually move outward, and the length of the month gets longer. In fact we know from growth scales in fossil corals (in which annual and monthly patterns can be discerned) that the day and month were both once substantially shorter than now: Earth has slowed down and Moon has moved farther away, as we predict.

Where will this process end? As the Earth's rotation slows down, the Earth will ultimately come to rotate *once per month* (the then-length of the month) and keep one face always towards the Moon. In this state, called "tidally locked," there is no drag on the tidal bulge: it will point exactly at the Moon as Earth and Moon rotate. (In fact, this corotation the current state the Moon is in with respect to the Earth.)

What will the ultimate length of the day (and month) be? Let $\omega_1 = 2\pi/(1 \text{ day})$, $\omega_2 = 2\pi/1$ month, $\omega_f = 2\pi/(\text{ultimate day and month})$. Then conservation of angular momentum gives (1 = Earth, 2 = Moon):

$$\begin{aligned}(\text{angular momentum now}) &= (\text{angular momentum then}) \\ I_1\omega_1 + I_2\omega_2 + \frac{M_1 M_2}{M}\omega_2 R^2 &= I_1\omega_f + I_2\omega_f + \frac{M_1 M_2}{M}\omega_f R_f^2\end{aligned}$$

while Kepler's law gives the future radius R' of the Moon's orbit,

$$\frac{R_f}{R} = \left(\frac{\omega_2}{\omega_f}\right)^{2/3}$$

to good approximation, since $M_1 \approx 81.3 M_2$, we have $M_1 M_2/M \approx M_2$. Also we can neglect the spin angular momentum of the Moon now (the Earth's alone being so much larger), and we can neglect *both* spins in the final state. (If you doubt these approximations, you can substitute back the answer we will get and check retroactively that they are justified.) Also we will use the moment of inertia of a *uniform* sphere, even though this is not quite a true assumption,

$$I_1 = \frac{2}{5} M_1 R_1^2 \ .$$

Then we have

$$\frac{2}{5} M_1 R_1^2 \omega_1 + M_2 \omega_2 R^2 = M_2 \omega_f R_f^2 = M_2 \omega_2 R^2 \left(\frac{\omega_2}{\omega_f}\right)^{1/3}$$

yielding

$$\begin{aligned} \frac{\omega_2}{\omega_f} &= \left[\frac{\frac{2}{5} M_1 R_1^2 \omega_1 + M_2 \omega_2 R^2}{M_2 \omega_2 R^2}\right]^3 = \left[1 + \frac{2}{5}\left(\frac{M_1}{M_2}\right)\left(\frac{R_1}{R}\right)^2\left(\frac{\omega_1}{\omega_2}\right)\right]^3 \\ &= \left[1 + \frac{2}{5} \times 81.3 \times \left(\frac{6400}{380000}\right)^2 \times 28\right]^3 = 1.99 \ . \end{aligned}$$

So the final length of day and month will be $28 \times 1.99 = 54$ days, and the final moon's orbital radius will be its current value times $(1.99)^{2/3}$ or about 590000 km. The current rate of lengthening of the day is about 20% per 10^9 years, so it will take $\gg 10^{10}$ yr for the process to go to completion. (By then, the Sun will have swollen to a giant and incinerated Earth and Moon anyway.)

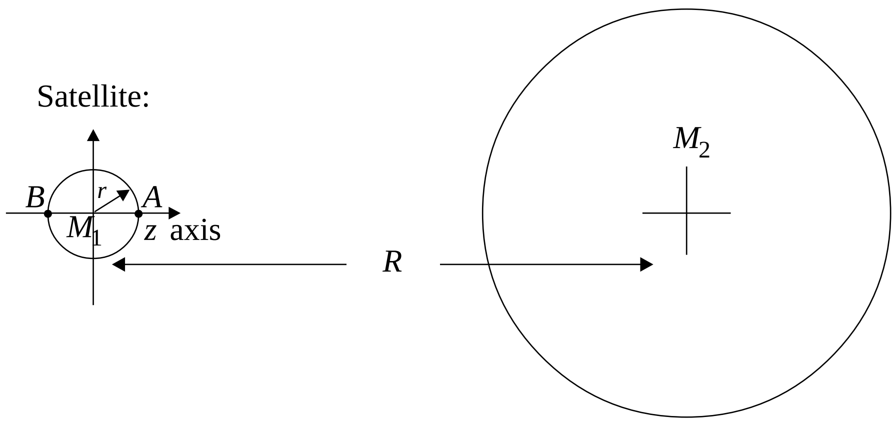
Incidently, although our specific example has been the Earth-Moon system, *exactly* these effects occur in close binary stars: Tidal drag causes them to come into "corotation," tidally locked to each other.

4.3.3 Roche stability limit for satellites

Let us now apply the previous result for the tidal potential (slight change in notion),

$$\Phi(r,\theta) = -\frac{GM_1}{r} - \frac{1}{2}\frac{Gr^3}{R^3}\left(3M_2\cos^2\theta + M_1\right) + \text{constant}$$

to the case where M_1 is a small satellite orbiting close to a large parent (star or planet) M_2, so $M_1 \ll M_2$.



It turns out that if R is too small, the satellite is torn apart by tidal forces. To see this, let's calculate the gravitational acceleration at points A and B in the figure. By symmetry it must be along the z axis, so

$$\begin{aligned} g_z &= -\nabla_z\Phi = -\frac{d}{dz}\left[-\frac{GM}{|z|} - \frac{1}{2}\frac{Gz^2}{R^3}(3M_2)\right] \\ &= -\frac{GM_1}{|z|^3}z + z\left(\frac{3GM_2}{R^3}\right) = Gz\left(-\frac{M_1}{|z|^3} + \frac{3M_2}{R^3}\right) . \end{aligned}$$

This is a *restoring* force only if the coefficient of z is negative. Otherwise the force is *away* from the center of the satellite and the satellite is torn apart, starting at its surface. Putting $|z| = r$ and considering a satellite of mean density ρ, so

$$M_1 = \frac{4}{3}\pi r^3\rho$$

we get the condition for *Roche stability*,

$$\rho > \frac{9}{4\pi}\frac{M_2}{R^3} \ .$$

Equivalently, no satellite of density ρ can be stable inside the Roche radius

$$R_{\rm crit} = \left(\frac{9M_2}{4\pi\rho}\right)^{1/3} \ .$$

If the parent body (e.g., a planet) has the same density as the satellite, then $M_2 = 4\pi R_2^3\rho/3$, and we get

$$R_{\rm crit} = 3^{1/3}R_2 = 1.44R_2 \ .$$

This is approximately what is going on with the rings of Saturn (and lesser rings of the other outer planets). Material inside the Roche limit cannot form into moons because of tidal forces from the parent body. ("Approximately" because our assumption of a spherical rather than deformed, satellite gives not quite the right numerical coefficient.)

4.3.4 Roche Lobe overflow

Let us go back (Section 4.3) to the basic formula for the tidal potential of two masses in the rotating CoM frame,

$$\Phi(\boldsymbol{r}) = -\frac{GM_1}{|\boldsymbol{r}-\boldsymbol{r}_1|} - \frac{GM_2}{|\boldsymbol{r}-\boldsymbol{r}_2|} - \frac{1}{2}\omega^2 r^2$$

with (Kepler)

$$\omega^2 = \frac{G(M_1+M_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|^3} \ .$$

A more carefully drawn version of the potential surface plot in 4.3 is the following contour plot:

The equipotentials Φ = const. for the Newtonian-plus-centrifugal potential in the orbital plane of a binary star system with a circular orbit. For the case shown here the stars have a mass ratio $M_1 : M_2 = 10 : 1$. The equipotentials are labelled by their values of Φ measured in units of $G(M_1 + M_2)/R$ where R is the separation of the centers of mass of the two stars. The innermost equipotential shown is the "Roche lobe" of each star. Inside each Roche lobe, but outside the stellar surface, the potential Φ is dominated by the "Coulomb" $(1/r)$ field of the star, so the equipotentials are nearly spheres. The potential Φ has local stationary points ($\nabla\Phi = 0$), called "Lagrange points," at the locations marked L_J.

The picture is drawn for the particular case $M_2/M_1 = 0.1$, but would be qualitatively similar for other mass ratios.

Focus attention on the contour marked "Roche lobe" that defines the *saddle point* between the two potential minima. (Look back at the surface plot in 4.3 if you need to visualize the "saddle point.") In a binary star system, each Roche lobe is the maximum size that its respective star can be. If (due to stellar evolutionary effects) one star tries to swell up to larger than the Roche lobe, it simply dumps mass through the "lip of the pitcher" (across the saddle point) onto the other star. This is called "Roche lobe overflow" and can profoundly affect the evolution of the binary star system.

For example, consider a case where M_1 is initially greater than M_2. Higher mass stars evolve faster, so M_1 may first swell up and dump mass onto M_2 (possibly also losing mass to infinity as a stellar wind). Eventually M_1 becomes a white dwarf star and starts to cool. Later, as M_2 evolves (now faster because it has gained mass) *it* may swell up and Roche-overflow back onto the white dwarf, a process that can be observed as an X-ray source (the X-rays being produced as the matter crushes down on the white dwarf surface). Finally the excess mass dropped onto the white dwarf may be too much for it to support, and it may collapse to a neutron star, in the process blowing off part of its mass as a supernova explosion!

Notice that the saddle point marked L_2 is a *very slightly* higher contour than L_1. Thus, a star that swells very slowly will dump preferentially onto its companion, but more rapid swelling can overflow both L_1 and L_2 simultaneously. Matter that goes over the "lip" L_2 is flung out to infinity in what would look like an expanding spiral trail (remember the coordinate system is rotating!). This is thought to be happening in so-called "violent mass transfer binaries" of which examples are W Serpentis and β Lyrae.

4.3.5 Effect of mass transfer on binary orbits

Suppose that M_1 is filling its Roche lobe and dumping mass onto M_2. This process conserves mass and angular momentum, but not energy (since the mass crashing down onto M_1 dissipates its energy into heat.

Since M_2 is gaining mass, we have

$$\dot{M}_2 = -\dot{M}_1 \geq 0 \ .$$

Note that the total mass M is constant.

Now write the angular momentum for a circular orbit,

$$L = \mu R^2 \Omega = \mu R^2 \left(\frac{GM}{R^3}\right)^{1/2} = \frac{M_1 M_2}{M^{1/2}} R^{1/2} G^{1/2} = (M_1 M_2 R^{1/2})(G^{1/2} M^{-1/2}) \, .$$

Now we conserve angular momentum:

$$0 = \frac{dL}{dt} = G^{1/2} M^{-1/2} \left(\dot{M}_1 M_2 R^{1/2} + M_1 \dot{M}_2 R^{1/2} + \frac{1}{2} M_1 M_2 \dot{R} R^{-1/2}\right) \, .$$

Solving for $\dot{R}$ and eliminating $\dot{M}_1$ in favor of $\dot{M}_2$ we get

$$\dot{R} = 2R \left(\frac{M_2 - M_1}{M_1 M_2}\right) \dot{M}_2 \, .$$

So, if the lighter star is losing mass, $M_1 < M_2$, then, since $\dot{M}_2 > 0$, we get $\dot{R} > 0$ and the stars draw gradually apart. Often this turns off the mass flow, since it puts M_1 deeper into its Roche lobe. Or, the mass flow proceeds only on the stellar evolutionary timescale that it takes M_1 to keep swelling to fill the ever-larger Roche lobe.

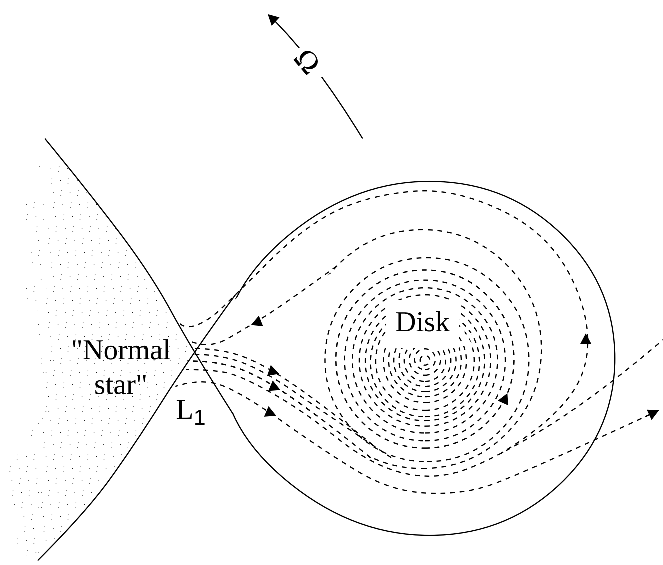
Things are different if the heavier star is losing mass, $M_1 > M_2$! Then $\dot{R}$ is negative. The mass flow causes the stars to get closer, which typically *increases* the mass flow, and so on in a catastrophic instability. This could theoretically end when the stars reach equal mass. But in practice, the mass flow is often so violent that hydrodynamic friction results in the stars' merging.

4.3.6 Accretion disks

Gas at rest at the "lip" L_1 of the Roche lobe has a specific angular momentum (that is, angular momentum per unit mass)

$$\tilde{L} = |\boldsymbol{r}_{L1} - \boldsymbol{r}_{\mathrm{CoM}}|^2 \, \Omega \, .$$

Referring to the figure in 4.3.4, you can see that $|\boldsymbol{r}_{L1} - \boldsymbol{r}_{\rm CoM}|$ can be significantly less than r (the 2-body separation). A consequence is that gas that flows over L_1 has only enough angular momentum to go into a circular orbit well inside the other Roche lobe — if it does not impact the surface of the other star first. If the other star is a white dwarf or neutron star, so-called "compact objects," it is small enough to allow the formation of an "accretion disk" of gas (see figure). Once in the accretion disk, gas slowly spirals into the compact object by the action of viscous forces (including magnetic and turbulent viscosity).



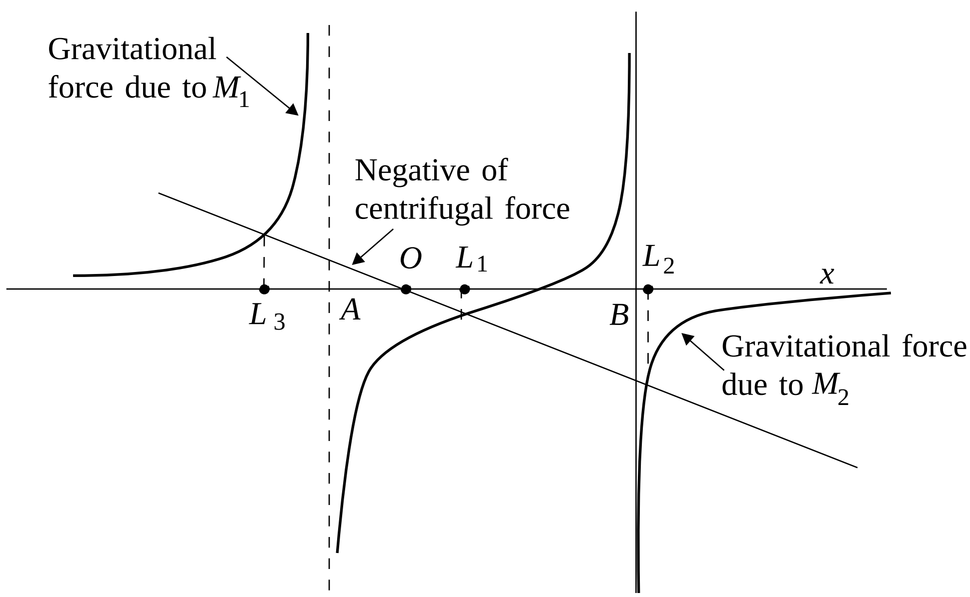
4.3.7 The Lagrange points

Referring again to the contour plot in 4.3.4, let us now take the masses to be safely tucked for inside their Roche lobes, as for the Earth and Moon, or Sun and Jupiter. We now ask the question: Are there any *equilibrium* points where a third orbiting test body could be placed, and where it would maintain a constant position in the rotating frame (that is, co-orbit with the same period as the two main bodies at constant position relative to them)?

The condition for such an equilibrium is that there be no acceleration at the chosen position, that is, the gradient of the potential Φ (including centrifugal term) must vanish,

$$\nabla\Phi = 0 \; .$$

Gradients vanish, in general, at extrema and at saddle points. In the contour plot you can see that there are 3 saddlepoints co-linear with masses M_1 and M_2. It is easy to see that there must always be these three (for any mass ratio) as the following graph illustrates:



You can see that gravitational force and (negative) centrifugal force must always cross at exactly three points, where

$$F_{\mathrm{grav}} = -F_{\mathrm{centrif}} \quad \text{or} \quad F_{\mathrm{grav}} + F_{\mathrm{centrif}} = 0 \; .$$

These three points, L_1, L_2, L_3 are the first three "Lagrange points" where a test mass can orbit. However, since they are saddle points (see contour picture) all three are *unstable*: if the mass is slightly perturbed, it falls into one of the potential wells or is flung off to infinity.

More interesting are the two maxima of the potential, labelled L_4 and L_5 in the contour plot. These are the so-called *stable Lagrange points*. Looking at the contour plot, or the surface plot at the beginning of 4.3, you can see that these maxima occur along the ridge-line of a long, banana-shaped ridge between the potential well of the combined masses and the centrifugal force potential that decreases toward infinity.

If a test mass is placed at L_4 or L_5, it will stay there. If it is perturbed slightly, it will execute (in the rotating frame) a stable, though not necessarily closed, orbit that goes around L_4 or L_5.

You might wonder how such an orbit can be stable if it is going around a potential *maximum*, not *minimum*. The answer is that, in the rotating frame, there is a Coriolis force $-\boldsymbol{\omega} \times \boldsymbol{v}$, so that a test particle's equation of motion is actually

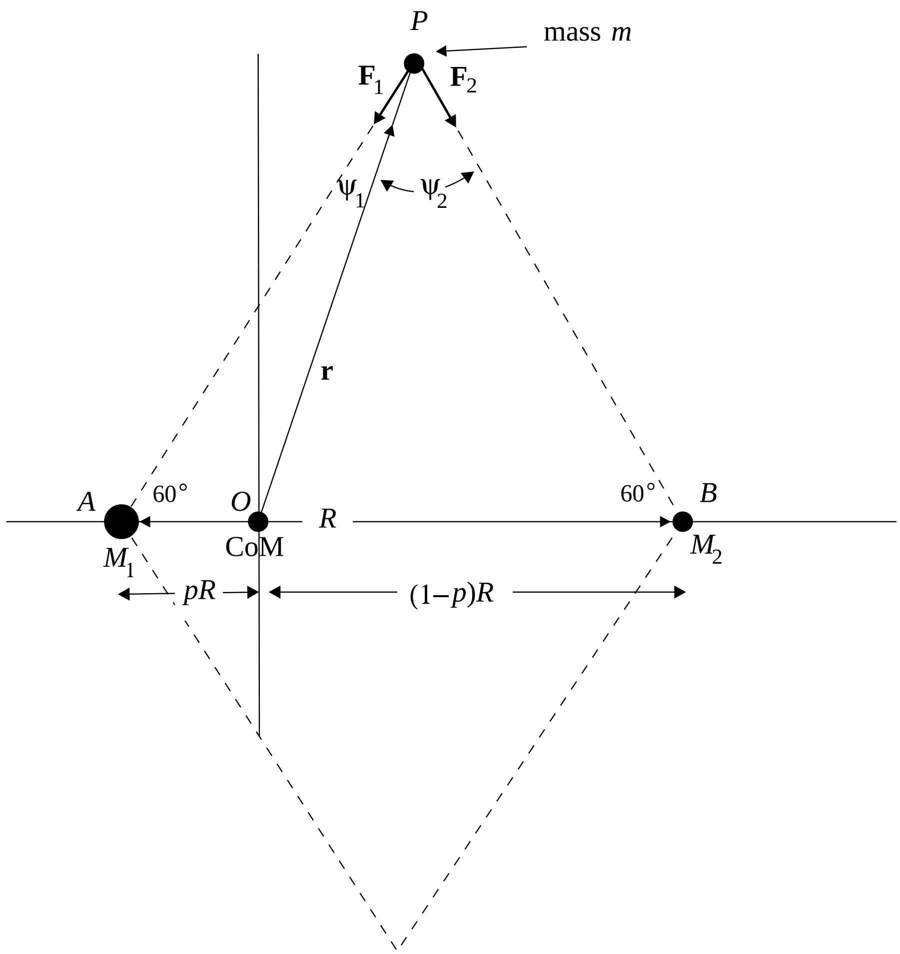
$$\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla}\Phi - \boldsymbol{\omega} \times \boldsymbol{v} \ .$$

If $\boldsymbol{\omega}$ in the contour picture is out of the page (M_2 and M_1 orbiting counter-clockwise), then a particle pushed "outward" from L_4 or L_5 experiences a clockwise force and goes into a small (stable) clockwise orbit around L_4 or L_5 respectively.

The best real-life example of objects orbiting a stable Lagrange point are the Trojan asteroids which orbit L_4 and L_5 of the Sun-Jupiter system. Between 1906 and 1908, four such asteroids were found: the number has now increased to several hundred (see figure). These asteroids are named for the heroes from Homer's *Iliad* and are collectively called the Trojans. Those that precede Jupiter (at L_4) are named for the Greek heroes (plus the

Trojan spy, Hektor), and those that follow Jupiter (at L_5) are named for the Trojan warriors along with the Greek spy, Patroclus. Some of the Trojans make complicated orbits taking as long as 140 years to meander around the "banana" of the potential (meanwhile, of course, orbiting the Sun every 11.86 years, just like Jupiter). [Abell 7$^{\text{th}}$ Ed., 18.3]

We have not yet determined the *location* of L_4 and L_5 in the figure. The amazing fact is that, independent of the mass ratio $M_1 : M_2$, L_4 and L_5 are exactly at the vertices of equilateral triangles formed with M_1 and M_2. This is so remarkable that we should at least verify it for fun:



We want to see that $\boldsymbol{F}_1 + \boldsymbol{F}_2$ exactly balances the centrifugal term $m\boldsymbol{r}\omega^2$. The magnitudes of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are

$$F_1 = \frac{GmM_1}{R^2} = \frac{Gm(1-p)M}{R^2} \qquad\qquad F_2 = \frac{GmM_2}{R^2} = \frac{GmpM}{R^2}$$

where $p \equiv M_2/M_1$ and $M = M_1 + M_2$. So the condition for force balance perpendicular to $\boldsymbol{r}$ is

$$F_1 \sin\psi_1 \stackrel{?}{=} F_2 \sin\psi_2 \ .$$

Apply the Law of Sines to the triangle AOP,

$$\frac{\sin 60°}{r} = \frac{\sqrt{3}/2}{r} = \frac{\sin\psi_1}{pR} \ .$$

So

$$\sin\psi_1 = \sqrt{3}pR/(2r)$$

and likewise using triangle BOP,

$$\sin\psi_2 = \sqrt{3}(1-p)R/(2r)\ .$$

Using the previous formulas for the magnitudes F_1 and F_2, the force check becomes

$$F_1\sin\psi_1 = \frac{\sqrt{3}pR}{2r}\frac{G(1-p)M}{R^2} \stackrel{?}{=} \frac{\sqrt{3}(1-p)R}{2r}\frac{GmpM}{R^2} = F_2\sin\psi_2\ .$$

Hey, it checks! So now we need to check the component *parallel* to $\boldsymbol{r}$ (which includes the centrifugal term):

$$F_1\cos\psi_1 + F_2\cos\psi_2 \stackrel{?}{=} mr\omega^2$$

Law of cosines on AOP:

$$\cos\psi_1 = \frac{r^2+R^2-p^2R^2}{2rR}$$

Law of cosines on BOP:

$$\cos\psi_2 = \frac{r^2+R^2-(1-p)^2R^2}{2rR}$$

Kepler's law:

$$\omega^2 = \frac{GM}{R^3}\ .$$

So

$$\begin{aligned}
\left[\frac{Gm(1-p)M}{R^2}\right]\left[\frac{r^2+R^2-p^2R^2}{2rR}\right] & \\
+\left[\frac{GmpM}{R^2}\right]\left[\frac{r^2+R^2-(1-p)^2R^2}{2rR}\right] &\stackrel{?}{=} \frac{GMmr}{R^3} \\
\frac{(1-p)(r^2+R^2-p^2R^2)+p[r^2+R^2-(1-p)^2R^2]}{2r} &\stackrel{?}{=} r \\
r^2+R^2(1-p+p^2) &\stackrel{?}{=} 2r^2\ .
\end{aligned}$$

Hmm. Is this true for all values of p? Yes! It is the Law of Cosines applied to the triangle BOP (using $\cos 60° = 1/2$):

$$\begin{aligned} r^2 &= R^2 + (1-p)^2 R^2 - 2(1-p)R^2 \cos 60° \\ &= R^2(1 - p + p^2) \ . \end{aligned}$$

So the force balance is exact for both components.

4.4 The virial theorem

Many-body (that is, more than 2-body) dynamics is something that comes later in the course, but we need to derive one important result, the *virial theorem*, now. Just for fun, here is a fairly fancy derivation, taught to me by George Rybicki. If this is above your level, just study it in a general way.

Suppose we have N bodies all interacting. Their kinetic energy is

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2$$

and their gravitational potential energy is

$$V = -\sum_{i \neq j} \sum_{j} \frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \ .$$

Newton's force law is

$$m_i \dot{\boldsymbol{v}}_i = -\left(\boldsymbol{\nabla} V\right)_{\boldsymbol{r}=\boldsymbol{r}_i} \ .$$

Define a quantity (a bit like a moment of inertia, but around a *point*, not an axis)

$$I \equiv \frac{1}{2} \sum_{i=1}^{N} m_i r_i^2 \ .$$

Differentiate with respect to time twice:

$$\dot{I} = \sum_i m_i \boldsymbol{v}_i \cdot \boldsymbol{r}_i$$

$$
\begin{aligned}
\ddot{I} &= \sum_i m_i \dot{\boldsymbol{v}}_i \cdot \boldsymbol{r}_i + \sum_i m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i \\
&= -\sum_i \boldsymbol{r}_i \cdot (\boldsymbol{\nabla} V)_{\boldsymbol{r}=\boldsymbol{r}_i} + 2T \ .
\end{aligned}
$$

Now here is the tricky part: The virial theorem comes about because the potential energy V has a *scaling relation* that describes how its numerical value would change if all positions $\boldsymbol{r}$ were stretched by some factor λ. In particular, you can see right away that

$$
V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2, \ldots, \lambda \boldsymbol{r}_N) = \frac{1}{\lambda} V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \ .
$$

Differentiating with respect to λ, and then setting $\lambda = 1$, we get

$$
\sum_i \boldsymbol{r}_i \cdot (\boldsymbol{\nabla} V)_{\boldsymbol{r}=\boldsymbol{r}_i} = -V \ .
$$

Thus

$$
\ddot{I} = V + 2T \ .
$$

This is called the time dependent virial theorem. It says, roughly, that if $V + 2T > 0$, the mean square size of the system must eventually be *increasing* while if $V + 2T < 0$ the size must eventually be *decreasing*. (Remember that V is negative, for this to make sense.)

It also follows, logically enough, that if a gravitating system is in equilibrium, neither increasing nor decreasing steadily in size, it must have

$$
\langle V + 2T \rangle = 0 \qquad \text{or} \qquad 2\langle T \rangle = -\langle V \rangle \ .
$$

Here the angle brackets denote the long-time *average*. To see this formally, average the time dependent theorem over a long time T:

$$
\langle V + 2T \rangle = \frac{1}{T} \int_0^T (V + 2T)\, dt = \frac{1}{T} \int_0^T \ddot{I}\, dt = \frac{1}{T} [\dot{I}(T) - \dot{I}(0)] \ .
$$

Now if all particles remain bounded with bounded velocities for all time (the definition of an equilibrium system) then $\dot{I}(t)$ remains bounded (see its formula above) and the right hand side goes to zero as $T \to \infty$, thus proving the desired time-averaged virial theorem.