Fast Operations Involving the Matrix $|t_i - t_j|$

George B. Rybicki

4 March 1998

1 Theory

This note is concerned with finding the inverse of the symmetric matrix $\boldsymbol{\Psi}$, with components

$$\Psi_{ij} = |t_i - t_j| \tag{1}$$

where the values t_i, $i = 1, 2, 3, \ldots, n$, are all distinct. This problem arises in the statistical treatment of time series, related to the "random walk" process.

We assume that the sequence t_i is given in strict ascending order, $t_1 < t_2 < \ldots t_n$ (if not true initially, this can be arranged by simple relabelling of the indices). Using the monotonicity of the t_i, the matrix can be written,

$$\Psi = \begin{pmatrix} 0 & t_2 - t_1 & t_3 - t_1 & \ldots & t_n - t_1 \\ t_2 - t_1 & 0 & t_3 - t_2 & \ldots & t_n - t_2 \\ t_3 - t_1 & t_3 - t_2 & 0 & \ldots & t_n - t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_n - t_1 & t_n - t_2 & t_n - t_3 & \ldots & 0 \end{pmatrix} \tag{2}$$

It can be shown that the inverse of Ψ may be expressed in the form

$$\Gamma = \Psi^{-1} = \mathbf{T} + \mu \mathbf{L}\mathbf{L}^T \tag{3}$$

where $\mathbf{T}$ is a symmetric tridiagonal matrix, and $\mathbf{L}$ is a column vector consisting of zeroes, except for the first and last elements, which are unity. In particular,

$$\mathbf{T} = \begin{pmatrix} d_1 & e_1 & 0 & \dots & 0 & 0 \\ e_1 & d_2 & e_2 & \dots & 0 & 0 \\ 0 & e_2 & d_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} & e_{n-1} \\ 0 & 0 & 0 & \dots & e_{n-1} & d_n \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{4}$$

where e_i and d_i are given by

$$e_i = \frac{1}{2}(t_{i+1} - t_i)^{-1}, \qquad 1 \le i \le n-1, \tag{5}$$

and

$$d_1 = -e_1 \tag{6}$$

$$d_i = -e_i - e_{i-1}, \qquad 2 \le i \le n-1, \tag{7}$$

$$d_n = -e_{n-1}, \tag{8}$$

and where μ is given by

$$\mu = \frac{1}{2}(t_n - t_1)^{-1}. \tag{9}$$

The inverse can also be written in the form

$$\Gamma = \Psi^{-1} = \begin{pmatrix} d_1 + \mu & e_1 & 0 & \dots & 0 & \mu \\ e_1 & d_2 & e_2 & \dots & 0 & 0 \\ 0 & e_2 & d_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} & e_{n-1} \\ \mu & 0 & 0 & \dots & e_{n-1} & d_n + \mu \end{pmatrix} \tag{10}$$

which is seen to be a tridiagonal matrix, modified by the addition of μ to each of the four "corner" elements.

We remark that these results can be trivially generalized to the case of the scaled matrix $S_{ij} = \alpha|t_i - t_j|$, where α is a constant. All elements of the inverse matrix (i.e., the d_i, e_i, and μ) are then simply scaled by the factor α^{-1}.

The above results, especially the representation (3) for the inverse of Ψ, permit a number of "fast," $O(n)$ matrix operations involving Ψ. We present three such operations:

1. There is a fast solution of the linear system

$$\mathbf{y} = \Psi\mathbf{x} \tag{11}$$

for $\mathbf{x}$ given $\mathbf{y}$, namely,

$$\mathbf{x} = (\mathbf{T} + \mu\mathbf{L}\mathbf{L}^T)\mathbf{y} = \mathbf{T}\mathbf{y} + \mu(\mathbf{L}^T\mathbf{y})\mathbf{L} \tag{12}$$

which involves only $O(n)$ operations.

2. There is a fast evaluation of the direct product $\mathbf{y} = \Psi\mathbf{x}$ itself, a kind of fast "convolution." In this case we write

$$\mathbf{y} = (\mathbf{T} + \mu\mathbf{L}\mathbf{L}^T)^{-1}\mathbf{x} \tag{13}$$

Using the Lemma in Appendix A, we have

$$\mathbf{y} = \mathbf{u} - \frac{\mu\mathbf{L}^T\mathbf{u}}{1 + \mu\mathbf{L}^T\mathbf{v}}\mathbf{v} \tag{14}$$

where

$$\mathbf{u} = \mathbf{T}^{-1}\mathbf{x}, \qquad \mathbf{v} = \mathbf{T}^{-1}\mathbf{L} \tag{15}$$

3. The following matrix evaluation is often required in practice,

$$\mathbf{w} = (\Psi + \mathbf{N})^{-1}\mathbf{y} \tag{16}$$

where $\mathbf{N}$ is a diagonal matrix and $\mathbf{y}$ is a column vector. This can also be written

$$\mathbf{w} = \mathbf{N}^{-1}(\mathbf{N}^{-1} + \Psi^{-1})^{-1}\Psi^{-1}\mathbf{y} \tag{17}$$

Noting that $(\mathbf{N}^{-1} + \Psi^{-1})^{-1}(\mathbf{N}^{-1} + \Psi^{-1}) = (\mathbf{N}^{-1} + \Psi^{-1})^{-1}\mathbf{N}^{-1} + (\mathbf{N}^{-1} + \Psi^{-1})^{-1}\Psi^{-1} = 1$, we have

$$\mathbf{w} = \mathbf{N}^{-1}\left[1 - (\mathbf{N}^{-1} + \Psi^{-1})^{-1}\mathbf{N}^{-1}\right]\mathbf{y} \tag{18}$$

or

$$\mathbf{w} = \mathbf{N}^{-1}\mathbf{y} - \mathbf{N}^{-1}(\mathbf{N}^{-1} + \mathbf{T} + \mu\mathbf{L}\mathbf{L}^T)^{-1}\mathbf{N}^{-1}\mathbf{y} \tag{19}$$

Since $\mathbf{N}^{-1} + \mathbf{T}$ is tridiagonal, this can be evaluated using the fast method of the Lemma in the Appendix.

A A Useful Lemma

Lemma 1 *Let* $\mathbf{T}$ *be a tridiagonal matrix and let* $\mathbf{U}$*,* $\mathbf{V}$*, and* $\mathbf{x}$ *be column vectors. Then*

$$(\mathbf{T}+\mathbf{U}\mathbf{V}^T)^{-1}\mathbf{x} = \mathbf{A} - \frac{\mathbf{V}^T\mathbf{A}}{a+\mathbf{V}^T\mathbf{B}}\mathbf{B}, \quad \text{where} \quad \mathbf{A} = \mathbf{T}^{-1}\mathbf{x}, \quad \mathbf{B} = \mathbf{T}^{-1}\mathbf{U} \tag{20}$$

which can be evaluated in $O(n)$ *operations.*

Proof The Sherman-Morrison formula is

$$(\mathbf{T}+\mathbf{U}\mathbf{V}^T)^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(1+\mathbf{V}^T\mathbf{T}^{-1}\mathbf{U})^{-1}\mathbf{V}^T\mathbf{T}^{-1} \tag{21}$$

Application of this immediately gives the stated result (20). Since this formula involves only two tridiagonal solutions and two scalar products, this is a "fast" process, involving only $O(n)$ operations. •